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DEPARTMENT OF FINANCE

Zhodnocení možností odhadu parametrů modelů vývoje cen investičních instrumentů

Assessment of Alternative Methods for Parameter Estimation of Investment Instruments Price Models

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1. Introduction
2. Characteristics of Investing and Investment Instruments
3. Methods for Estimating the Model Parameters
4. Application of Selected Methods and Their Evaluation
5. Conclusion

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Declaration of Utilization of Results from the Diploma Thesis
List of Annexes
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The declaration

“Herewith I declare that I elaborated the entire thesis, including all annexes, independently.”

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1 Introduction

The current economy is dynamic and is constantly changing being driven by capital flows. The demand for finances is much larger than the available resources to individual entities which leads to the use of different investment instruments that for the issuers represent a source of new finances and for investors an investment opportunity that can generate profit. The growing amount of investment possibilities and the turbulent financial environment cause the investment decision-making process to become more difficult. One of the ways to determine the right investment instrument is the estimation of the future price evolution. Accurate estimation represents a powerful tool for the investment decision-making process.

The prediction of the price evolution is a complex and demanding task and has to be based on other investment instruments' analyses. Even though the prediction should represent just an additional information, the right estimation can be an informational advantage and in conclusion help gaining higher returns. Since the increase of returns is, without a doubt, the main objective for the majority of investors, many of them focus on the improvement of the price estimation techniques. The assessment of different methods of the estimation of price evolution was the objective of the author's bachelor thesis. Since this field is very complex and from the conclusion of the bachelor thesis many possibilities for further work have arisen, the issue of price estimation will be examined from a more detailed perspective in this diploma thesis.

In the price estimation process the most attention should be given to the volatility parameter, because it represents the instruments' risk, i.e. the main source of uncertainty. Correct construction of the volatility could significantly improve the estimation and lead to higher returns. How great of an impact do different approaches of volatility construction have on the price estimation will be explored in this thesis.

The objective of this diploma thesis is the assessment of alternative methods for volatility estimation of selected investment instruments which will be applied to the Monte Carlo simulation with geometric Brownian motion.

This diploma thesis is divided, apart from the introduction and the conclusion, into three chapters.

In the first chapter of the thesis the attention is focused on the theoretical background of the investment process. The investment, investment instruments, financial systems and the players that engage in the financial markets are discussed in this section. There is also described the investment process, criteria for investment decisions and financial models.

The second chapter represents the methodical section of the thesis. In this chapter four approaches of volatility construction are described, namely, *the crisp volatility (M1), the box approach (M2)*, *the fuzzy partition approach (M3)* and *the fuzzy transform approach (M4)*. Also the selected price model, the Monte Carlo simulation with geometrical Brownian motion, to which all approaches of volatility construction will be applied, is defined.

The final chapter of the thesis represents the application section. Selected methods are applied to three different investment instruments and the calculations are based on two different ranges of historical data time series. The chapter concludes with assessment of obtained results.

2 Characteristics of Investing and Investment Instruments

Investments represent a vital part of modern economies. The current economies need enormous amounts of *investments* to be able to produce the goods and services demanded by consumers.

The positive impact of the investments is that they help to increase the productivity of labour. However, investments usually require huge amounts of funds, much bigger than resources available to a single individual or institution. The solution can be found by selling financial claims (i.e. investment instruments like stock, bonds etc.) using the *financial markets*, where households, businesses, and governments can quickly raise large amounts of funds from accumulated savings. The entity performing the investment then hopes to repay its loans from the financial marketplace by generating future income.

The transformation of savings into investments and the exchange of current income for future income, so that production, income, and employment can grow, are enabled by the *money* and *capital markets*.

These financial markets represent the heart of the global financial system. Their role in channelling savings into investment is absolutely crucial to the health of the economy. This transformation accelerates the economy's growth and development of new businesses and new jobs (Rose, et al., 2008).

The following subchapters will be focused on the fundamentals of investing, investment instruments, financial markets, investment process and financial modelling.

2.1 Investment

An *investment* can be described as the current commitment of money or other resources in the expectation of gaining future benefits. Investment indicates the idea that safety of principal is important; by contrast, *speculation* is far riskier (Bodie, et al., 2011).

It is important to distinguish investment and speculation, as these two terms can be mistaken at times. There is a very fine line between them. Speculation needs an investment and investments are somewhat speculative in nature. Even some financial experts have called investment a well-grounded and carefully planned speculation or that a good investment represents a successful speculation.

The term *investment* generally represents a commitment that is relatively free from certain risks of loss. It is limited to situations that promise dependable income, relatively stable value, a modest rate of return and relatively little chance for outstanding capital appreciation. Investors who seek high-income returns or large capital gains have to exchange investment for speculation.

Speculation differs from investment in the time limit of investing and the risk-return characteristics of the investment. Even though an investor is usually interested in a good and consistent rate of return for a long period of time, a speculator is more interested in gaining very large returns in a short time period (Bhat, 2008).

2.2 Investment Instruments

The material wealth of a society is determined by the productive capacity of its economy, i.e. the goods and services that its members can create. This capacity represents a function of **real assets** of the economy: the land, real estate, machines, and knowledge that can be used to produce goods and services. In contrast **financial assets**, such as stock and bonds, are no more than a sheet of paper or a computer entry. They do not contribute to the productive capacity of the economy directly. These assets are the tools with which individuals in well-developed economies hold their claims on real assets. Financial assets represent the claims to the income generated by real assets. While on the one hand real assets generate net income to the economy, on the other hand financial assets simply define the allocation of income or wealth among investors (Bodie, et al., 2011).

2.2.1 Real Assets

Real assets represent such tangible assets such as real estate, machinery, cars, or airplanes but also precious stones, precious art objects and metals. Many of these assets are usually held by operating companies, such as real estate developers, airplane leasing companies, or manufacturers. Numerous institutional investment managers, however, have been adding real assets to their portfolios as direct investments (i.e. direct ownership of real assets) and indirect investments (i.e. indirect ownership of real assets, e.g. purchase of securities of companies that invest in real assets or real estate investment trusts). The attractiveness of investments in real assets lies in the income and tax benefits that are often generated, and also in the fact that the value of real assets may have a low correlation with other investments that investors hold.

Direct investments in real assets usually require significant management to guarantee that the assets are maintained and used efficiently. Investing in such assets sometimes requires hiring personnel to manage these assets, and it can become quite a costly matter.

Real assets have unique properties in the sense that no two assets are alike. Real assets usually differ in their conditions, locations, and suitability for various purposes. These differences are significant for their owners, thus the market for a given real asset may be very limited. For this reason real assets tend to trade in very illiquid markets.

The heterogeneity, illiquidity and substantial costs of managing real assets represent factors that complicate the valuation of real assets and usually make them unsuitable for most investment portfolios. However, because these problems often cause real assets to be misvalued in the market, information-motivated traders may occasionally identify significantly undervalued assets. Still, the benefits from purchasing such assets can be often offset by the substantial costs of searching for them and managing them.

Another option for investors are *indirect investments in real assets*. For these purposes investors can use entities, such as real estate investment trusts (REITs) and master limited partnerships (MLPs) that securitise real assets and facilitate indirect investments in real assets. These securities tend to trade in more liquid markets because they are much more homogenous and divisible than real assets that they represent. These securities represent a much more suitable investment then the real assets themselves.

Investors that seek exposure to real assets also have the opportunity to buy shares in corporations that hold and operate real assets. Although nearly all corporations hold and operate real assets, many of them specialise in assets that are particularly attractive to investors that are seeking exposure to specific real asset classes. For example, an investor that is interested in owning aircraft can buy an aircraft leasing company (McMillan, et al., 2011).

2.2.2 Financial Assets

Financial assets are commonly classified into three broad types (Bodie, et al., 2011):

- Fixed-income;
- Equity;
- Derivatives.

From **fixed-income** or **debt securities** can be generated either fixed stream of income or a stream of income determined by a specified formula. These securities come in a tremendous variety of maturities and payment provisions. *Money market* refers to short-term debt securities that are highly marketable, and generally of a very low risk, for example U.S. Treasury bills or bank certificates of deposit (CDs). In contrast, *capital market* includes in fixed-income securities long-term securities such as Treasury bonds, as well as bonds issued by local municipalities, and corporations. In terms of default risk, these bonds range from very safe (e.g. Treasury securities) to relatively risky (for example, high-yield or "junk" bonds).

Equity, or common stock, in a firm represents an ownership share in the corporation. Equity holders are not promised any specific payment. They receive dividends that the firm may pay and have a prorated ownership in the real assets of the firm. The performance of equity investments is directly connected with the success of the firm and its real assets. If the firm is successful, the value of equity will increase, and vice versa. Because of that, equity investments tend to be *riskier* than investments in debt securities.

Derivative securities such as forwards, options, swaps and futures contracts provide payoffs that are determined by the prices of *other* assets, e.g. bond or stock prices. The name of these securities originates from the fact that their values derive from the prices of other assets. Derivatives have become an important part of the investment environment. One of the major uses of derivatives is to hedge risks or transfer them to other parties.

2.3 Mutual Funds

A **mutual fund** represents a subject of collective investment. It is founded and operated by an *investment company* that raises money from shareholders and invests it in stocks, bonds, options, futures, currencies, or money market securities. The mutual fund is not an independent legal entity. Finances are concentrated in the fund by the issue of *unit certificates* (hereinafter just units). In **open-end mutual funds** the issuing of units is not restricted, and the investment company is obliged to buy the units back from the shareholders. These units are not traded on public markets. In **close-end mutual funds** the issue and sale of units are limited in quantity and time. A shareholder does not have a right for units repurchase by the investment company and units are traded on public markets (Fialová, & Fiala, 2009). By buying a unit in mutual funds investors get advantages of diversification and professional management and all of them share equally all gains and losses generated by the fund (Downes, & Goodman, 2010).

There are dozens of different types of mutual funds. Investors should assess their degree of risk tolerance before they choose the appropriate fund. Funds are generally classified by an investment policy into one of the following groups (Downes, & Goodman, 2010):

- **Money market funds** invest in money market securities such as commercial paper or certificates of deposits. The average maturity of these assets is usually a bit more than one month.
- **Equity funds** invest mainly in stock, although they may also hold fixed-income or other types of securities.
- **Sector funds** concentrate on a particular industry such as biotechnology, utilities, precious metals, telecommunications or securities of particular countries.
- **Bond funds** specialise in the fixed-income sector. Within this sector there is considerable room for further specialization. For example, funds concentrate on corporate bonds, Treasury bonds or municipal bonds. Numerous funds also specialise by maturity, ranging from short-term to intermediate to long-term, or by the credit risk of the issuer, ranging from very safe to high-yield, or "junk" bonds.
- **International funds.** Numerous funds have international focus. *Global funds* invest in securities worldwide. *International funds* invest in securities of firms located outside the home country. *Regional funds* focus on a particular part of the world, and *emerging markets funds* invest in companies of developing countries.
- **Balanced funds** hold both equities and fixed-income securities in relatively stable proportions. They are designed to be an individual's entire investment portfolio. In *life-cycle funds* asset mix can range from aggressive (for younger investors) to conservative (for older investors). In *static allocation life-cycle funds* mix across stocks and bonds remains stable whereas in *targeted-maturity funds* mix gradually becomes more conservative as the investor ages.
- **Asset allocation and flexible funds** hold both stocks and bonds. These funds may dramatically vary based on the proportions allocated to each market in accord with the portfolio manager's forecast of the relative performance of each sector.

Therefore these funds are engaged in market timing and are not designed to be low-risk investment instruments.

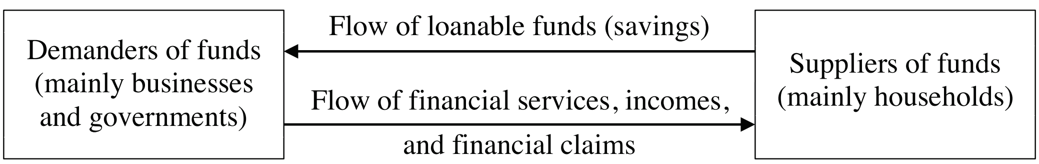
- **Index funds** try to match the performance of a broad market index. Shares and their proportion to each securities representation originates from securities included in a particular index, e.g. S&P 500. Index funds represent a low-cost way for small investors to pursue a passive investment strategy, i.e. to invest without performing security analysis.

2.4 Financial Systems

Markets provide an indispensable conduit for the transformation of savings into **investments**. This action accelerates the economy's growth and development of new businesses and jobs. The aim of this subchapter is to describe the financial system and the markets therein.

Financial system represents a collection of markets, institutions, laws, regulations and techniques through which bonds, stocks and other securities are traded, interest rates are determined, and financial services are produced and delivered around the world. The primary task of a financial system is to move loanable funds from those who save to those who borrow to buy goods and services and to invest so that the global economy can grow and increase the standard of living. The financial system determines both the cost and the quantity of funds available in the economy. The *global financial system*, depicted in Figure 2.1, is an essential part of the *global economy system*. It is not possible to understand one of these systems without understanding the other.

Figure 2.1 The Global Financial System



Source: Rose, P., Marquis, M., 2008, p. 7.

The *global economy system* generates a flow of production in return for a flow of payments. The task of allocating resources is most often transferred upon *markets*. **Market** represents an institution through which buyers and sellers meet to exchange goods, services and productive resources. This process of exchange determines which goods and services will be produced and in what quantity. In the global economy system, there are basically three types of markets:

- Factor markets;
- Product markets;
- Financial markets.

Financial markets have a vital function within the *global economy system*. They channel savings to those individuals and institutions that need more funds for spending then they have from their current incomes. Financial markets represent the heart of the global financial system. They attract and allocate savings and set interest rates and the prices of investment instruments.

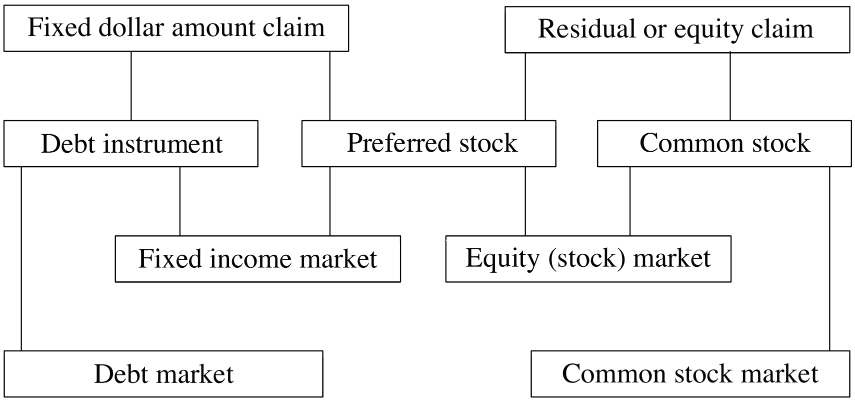
Many roles of the *global financial system* are also fulfilled through *financial markets*. They can be viewed as channels through which a vast flow of loanable funds moves; a flow that is continually being drawn upon by demanders of funds and continually being replenished by suppliers of funds (Rose, & Marquis, 2008).

2.4.1 Classification of Financial Markets

The flow of funds around the world may be divided into various segments, depending on the characteristics of financial claims being traded and the needs of different investors (Rose, & Marquis, 2008).

Based on the *type of financial claim* financial markets can be divided into **debt market** and **stock market**. Debt instruments are traded on debt markets. Equity instruments are traded on equity market. This market is alternatively called stock market. However, preferred stock represents an equity claim that entitles the investor to obtain a fixed dollar amount. Therefore preferred stock shares are classified as part of both debt and equity market. Generally, debt instruments and preferred stock are classified as part of the **fixed-income market** and stock market excluding preferred stock is called **the common stock market**, see Figure 2.2 (Fabozzi, & Modigliani, 2009).

Figure 2.2 Classification of Financial Markets by Type of Claim



Source: Fabozzi, F., Modigliani, F., 2009, p. 10.

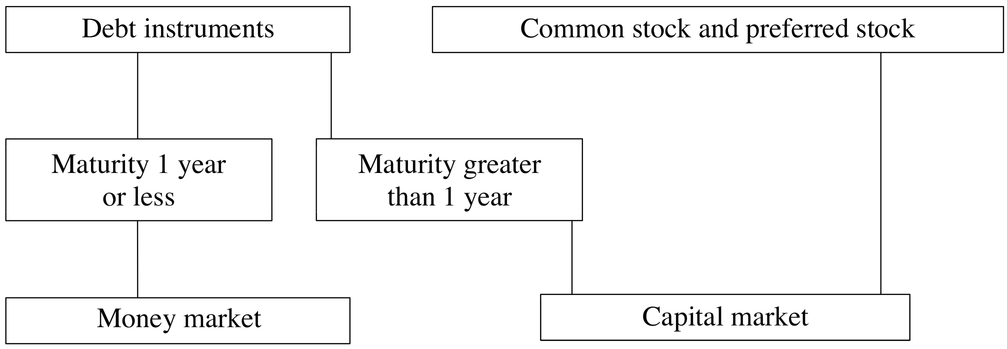
One of the most important classifications of the financial markets is by the *maturity of the claims* to **money market** and **capital market** (see Figure 2.3).

Money market is a financial market for short-term investment instruments, i.e. instruments with maturities of at most one year. Money market represents an institution through which individuals and institutions with *temporary* surpluses of funds meet the needs of borrowers who have *temporary* funds deficits. Therefore, the money market allows economic units to manage their liquidity positions. Principal functions of the money market are to finance the working capital needs of corporations, providing governments with short-term funds instead of tax collections, and supply funds for speculative buying of securities and commodities.

Capital market is a financial market for longer maturity investment instruments, i.e. instruments with maturities of more than one year. This market is designed to finance long-term investments by businesses, governments, and households. Thanks to the trading of funds in the capital market it is possible to construct financially demanding projects, such as factories, highways, schools, and homes.

Since the cutoff between short-term and long-term assets is the maturity of one year, the debt market can be divided into debt instruments that are part of the money market, and those that are part of the capital market, depending on the maturity. Since equity instruments are generally perpetual, they are classified as part of the capital market (see Figure below).

Figure 2.3 Classification of Financial Markets by Maturity of Claim



Source: Fabozzi, F., Modigliani, F., 2009, p. 11.

Another possible way to classify financial markets is by *whether the financial claims are newly issued*. The **primary market** is for the trading of newly issued investment instruments. Its chief function is raising financial capital to fund new investments in equipment, buildings, and inventories. In contrast, the market for investment instruments previously issued is referred to as the **secondary market**. Its principal function is to provide liquidity to investors. This market does not support new investments. The volume of trading in the secondary market is much larger than in the primary market. Still, both markets are closely intertwined (Fabozzi, & Modigliani, 2009).

It is also possible to distinguish *spot markets*, *futures* or *forward markets*, and *options markets*. On a **spot market** assets are traded for immediate delivery (usually within one or two business days). A **futures** or **forward market** is designed to trade contracts calling for the future delivery of investment instruments. Finally, through **option markets** are traded contracts that give an investor the right to either buy designated instruments from or sell designated instruments to the writer of the option at a guaranteed price at any time during the life of the contract. Through options it is possible to lock in prices of assets no matter which way those prices move before the options expire. Future, forward and option markets offer investors in both money and capital markets an opportunity to reduce risk (Rose, & Marquis, 2008).

2.4.2 The Players

There are three major players in the financial markets:

- **Firms** represent net borrowers. They raise capital to pay for investments. The income generated from investments provides the returns to investors who purchase the investment instruments issued by the firm.
- **Households** are usually net savers. They buy the investment instruments issued by firms that need to raise funds.
- **Governments** can be borrowers or lenders. It depends on the relationship between tax revenue and government expenditures.

Corporations and governments do not sell all of their investment instruments directly to individuals. For example, approximately half of all stock is held by large financial institutions such as mutual funds, banks, and pension funds. These financial institutions stand between the investment instrument issuer (the firm) and the ultimate owner of the instrument (the individual investor). For this reason, they are called financial intermediaries (Bodie, et al., 2011).

Financial intermediaries help entities to reach their financial goals. These intermediaries are commercial, mortgage, and investment banks; credit unions, credit card companies, and various other finance corporations; mutual funds and hedge funds; and insurance companies. Financial intermediaries help their clients to solve the financial problems that they face more efficiently than they could do so by themselves. They are vital to well-functioning financial systems. Financial intermediaries, and the types of services that they provide, are as follows (McMillan, et al., 2011):

- Brokers, Exchanges, and Alternative Trading Systems;
- Dealers;
- Securitizers;
- Depository Institutions and Other Financial Corporations;
- Insurance Companies;
- Arbitrageurs;
- Settlement and Custodial Services.

2.5 The Investment Process

One of the biggest investment challenges faced by individuals and institutions is how to allocate funds for future needs. Irrespective of the ultimate goal, they all face challenges that extend beyond just the choice of what asset classes to invest in. They have to decide whether to invest in individual securities, evaluating each in isolation, or should they take a portfolio approach. Using this approach the individual securities are evaluated in relation to their contribution to the investment characteristic of the whole portfolio.[1] The benefits of diversified portfolio will be explained in the next part of this subchapter (Bodie, et al., 2011).

2.5.1 A Portfolio Viewpoint on Investing

Portfolio *diversification* helps investors spread away some of the risk associated with unexpected or radical fall of the price of particular investment instrument, as it happened with the famous case of Enron Corporation in 2002.

In addition to avoiding potential disaster that is caused by overinvesting in a single security, portfolios also generally offer equivalent expected returns with *lower overall volatility* of returns, thus lower risk.

A major reason why portfolios can effectively reduce risk is that combining investment instruments whose returns are *negatively correlated* provides desirable diversification. The fact that these returns may offset each other at the same time creates the potential diversification benefit that is attributed to portfolios (McMillan, et al., 2011).

2.5.2 The Portfolio Management Process

The investment process has three steps that are critical in the establishment and management of an investment portfolio (McMillan, et al., 2011):

- **The Planning Step**
    - Understanding the investor's needs;
    - Preparation of an investment policy statement (IPS).
- **The Execution Step**
    - Asset allocation;
    - Security analysis;
    - Portfolio Construction.

[1] An investor's portfolio represents a collection of investment instruments.

- **The Feedback Step**
  - Portfolio monitoring and rebalancing;
  - Performance measurement and reporting.

The first step in the investment process, **the planning step**, helps to understand *investor's needs*, and to develop the *investment policy statement* (IPS). The IPS is a written planning document that defines the investor's investment objectives and the constraints that apply to the portfolio. The IPS can be used as a benchmark (e.g. specific rate of return, performance of a specific market index) for the feedback stage to assess the performance of the investments and whether objectives have been met. It is important to review and update the IPS regularly (e.g. every three years or when a major change in the investor's objectives, constraints, or circumstances occurs).

In **the execution step**, a suitable portfolio based on the IPS is constructed. This step begins with deciding on a target *asset allocation*, which defines the weighting of asset classes to be included in the portfolio. After this step follows the *security analysis* and *portfolio construction*. Each of these steps will be described below.

Asset allocation is focused on assessing the risk and return characteristics of the available investments. The analyst or investor himself forms economic and capital market expectations that can be used to form a suggested allocation of asset classes suitable for the investor. Decisions that must be made in this step include the distribution of equities, fixed-income securities and cash; sub-asset classes, such as corporate and government bonds; and geographical weightings within asset classes. Alternative assets (e.g. real estate, commodities and hedge funds) can also be included.

Security analysis is focused on identifying attractive investments in particular market sectors. For the purpose of security analysis, detailed knowledge of the companies and industries they cover allows assigning a valuation of an investment instrument regarding their future behaviour; expected return, associated risk and most of all allows identifying preferred investments (McMillan, el al., 2011).

For valuation of investment instruments, first of all, it is required to understand the characteristics of the different assets to estimate their worth. Secondly, a valuation model has to be applied to these investment instruments to estimate the value (or price). The relative attractiveness of the investment instrument can be determined by comparing the estimated value with the current market price of the investment instrument. There are two approaches to security analysis, *fundamental analysis* and *technical analysis* (Dash, 2009).

Fundamental analysis is based on a statement that the intrinsic value of an asset is equal to the present value of cash flows or earnings that asset holder is expected to receive. The analyst has to forecast the earnings and dividends expected from the stock and then convert them to their equivalent present value by using an appropriate discount rate that reflects the riskiness of that stock. This way the intrinsic value of a stock is determined and it can be compared to the current market price of that stock. Stocks that have higher intrinsic value than the current market price are perceived as undervalued or under priced and should be purchased. In the opposite case, the stock is perceived as over-valued or over priced and should be sold off. However, fundamental analysts believe that the market itself will correct this mis-pricing of investment instruments in the future, i.e. under priced stocks will show unusual appreciation while over priced stocks will show unusual depreciation.

Technical analysis is focused on the study of historical stock market prices of a firm in order to predict the future price movements. By identifying an emerging trend or a pattern in price movements of the stock, the technical analyst, often called a chartist, tries to predict accurately the future price movements of the particular stock.

While performing the security analysis of an investment instrument, the possibility to ***predict its price evolution*** represents valuable information for investors. Even though it is vital to perform analyses mentioned above, and their results should be the foundation for further prediction, the estimation of price evolution represents a useful tool in the investment process. It can offer additional information, which can help investors not only in deciding whether or not to invest in a specific instrument, but also in their comparison. Since the investment decision-making process is a complicated and broad field, all information that can reveal the likely evolution of the instruments characteristics plays an important role in this process. It is this part of the investing process that will be the main focus of this diploma thesis.

Portfolio construction represents the final stage of the execution step. When constructing a portfolio, the target asset allocation, security analysis, and investor's requirements (as set out in the IPS) should be taken into consideration. The key objective is to achieve the benefits of diversification. Decisions must be taken on weightings of asset class, sector within an asset class, and the selection and weighting of individual investment instruments. The relative importance of these decisions on portfolio performance partly depends on the selected investment strategy. Even though all decisions have an effect on portfolio performance, the asset allocation decision is commonly viewed as having the greatest impact. An important part of the portfolio construction process represents the risk management. The investor's risk tolerance is set out in the IPS, and it should be made certain that the portfolio is consistent with it. The diversified portfolio perspective should be applied: important is not the risk of any single investment, but rather how all the investments perform as a portfolio. The final stage of the portfolio construction involves trading. Once the decision on which investment instruments to buy and in what amounts is made, the instruments must be purchased.

Finally, **the feedback step** is used to rebalance the portfolio due to a change in, e.g. market conditions or circumstances of the investor. This step consists of *portfolio monitoring and rebalancing* and also *performance measurement and reporting*.

Once the portfolio has been created, *portfolio monitoring and rebalancing* has to take place. The composition of the portfolio has to be revised, due to the changes in security prices and fundamental factors. When as a result of market movement instruments weightings have changed compared to the desirable levels, rebalancing may also be required. Revision of the portfolio may also be required if the investor's needs or circumstances have changed.

Performance measurement and reporting is the last step in the portfolio management process. The performance of the portfolio must be measured to allow the assessment of whether the investor's objectives have been met. Based on the performance analysis results it may be suggested that it is necessary to review the investor's objectives and perhaps even change the IPS.

2.6 Criteria for Investment Decision

In investment decision-making process, the foremost and most important criterion is without a doubt safety. This criterion can be measured by the risk of specific investment instrument or portfolio.

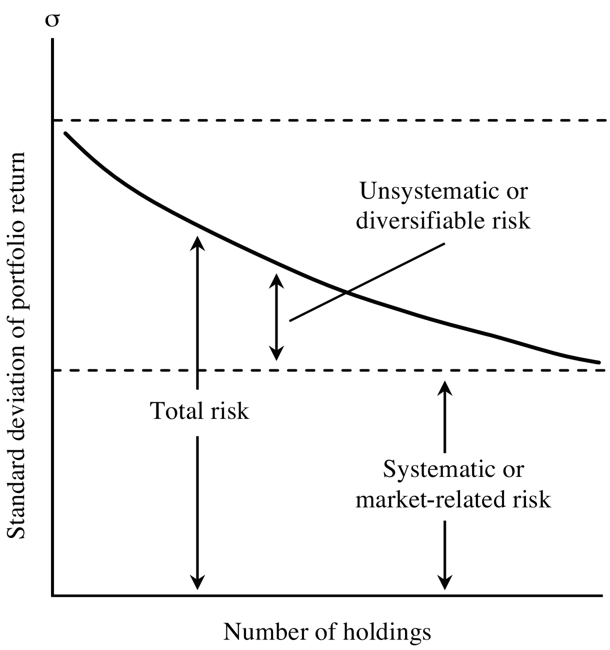
Risk can be described as the probability that the expected return from the investment instrument will not materialise. All investments involve uncertainties that make future investment returns risk-prone. These uncertainties can be caused by the political, economic and industry factors (Bhat, 2008).

Modern investment analysis classifies the sources of risk that cause variability in return into two general types (McMillan, et al., 2011):

- **Systematic risk** – represents the risk that is inherent in the overall market and cannot be avoided. This risk is non-diversifiable as it affects the market as a whole. It is pervasive in nature, such as market risk, interest rate risk, economic cycles, political uncertainty, and natural disasters (see Figure 2.4).
- **Unsystematic risk** – represents a risk that is local or limited to a particular instrument, industry, individual company (e.g. business or financial risk), that will not affect assets outside of that asset class. Investors can avoid unsystematic risk through diversification by constructing a portfolio of investment instruments that are not highly correlated with one another (see Figure 2.4).

The **total variance** of the investment instrument or portfolio equals the sum of *systematic variance* and *unsystematic variance*. Although there are frequent references to total risk as the sum of systematic risk and unsystematic risk, in those cases, the statements refer to variance, not standard deviation.

Figure 2.4 Systematic and Unsystematic Portfolio Risk



Source: Fabozzi J., Modigliani F., 2009, p. 153.

Even though risk is the most important criteria, it always has to be assessed along with other significant characteristics of investment instruments. The criteria that must be taken in consideration are *liquidity*, *return* and *maturity*.

Liquidity of an investment instrument refers to the ability of the investor to convert it into cash on short notice without incurring any loss. If an investment instrument is held till maturity it will give a definite return, while if sold prematurely the risk of loss is high.

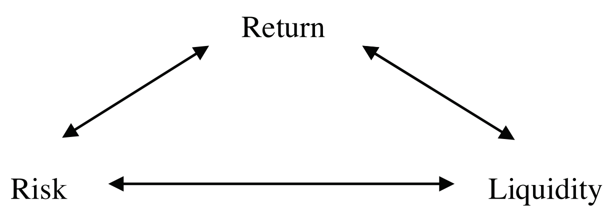
Return of an investment instrument represents the return earned from it by way of interest, dividend, and capital appreciation. Some investment instruments do not pay interest and are redeemed at face value.

The majority of empirical research analyses *returns to investors* rather than prices. Returns are more appropriate for a number of reasons. The most important is that returns, unlike prices, are just weakly correlated through time (Taylor, 2011).

For this reason all performed calculations in this thesis will be based on returns of selected investment instruments. The technique of converting the time series of prices into time series of returns will be discussed in subchapter 3.1.2.

The decision-making process is based on the relation of these three criteria. As it can be seen from the Figure 2.5, it is impossible to achieve maximum values within all three criteria at once. Investor has to choose whether he prefers safety, then he has two expect lower return, or he is willing to undertake higher risk, so that he can reach higher return. Higher liquidity is associated with lower return and vice versa, low liquidity of an investment instrument should be compensated by a higher return.

Figure 2.5 Risk, Liquidity and Return Triangle



Maturity represents the life of an instrument. Some investment instruments have fixed original maturities whereas others can have tailor-made maturity like certificate of deposit. The general rule states that the longer the maturity, the greater the return (Dash, 2009).

The majority of investment instruments are denominated in one **currency**, such as Czech koruna (CZK) or euro (EUR). Investors have to choose them with that feature in mind. To reduce *foreign exchange risk*, some issuers have issued dual currency securities, e.g. some pay interests in one currency but principal or redemption value in second. Some bonds carry a currency option that allows the investor to choose whether the payments of either interest or principal will be made in either of those two currencies. Since the change of the chosen currency rate represents a risk that can greatly influence the return of an investment instrument, while investing in foreign currencies it is vital to observe the evolution of currency rate and take it into account while selecting an investment instrument. The right selection can additionally increase the return of the selected investment instrument (Fabozzi, et al., 2009).

2.7 Volatility

A characteristic that measures the risk of an investment instrument, i.e. the characteristic of an investment instrument to rise or fall sharply in price within a short-term period, is called **volatility** (Downes, & Goodman, 2010).

A volatility of return denotes the rate at which prices change. This rate is defined by standard deviation of the instruments returns (see subchapter 3.1.1). The volatility can also be described as a standard deviation of the change in the logarithm of a price during a stated period of time. Risk managers are particularly interested in measuring and predicting volatility, as higher levels imply a higher chance of a large adverse price change.

There are different types of volatility, such as *historical volatility*, *conditional volatility, stochastic volatility* and *implied volatility*.

Historical volatility (also called *realised volatility*) is a standard deviation of a set of *previous* returns. This type of volatility will be described in subchapter 3.1.1.

Conditional volatility represents a standard deviation of a *future* return that is conditional on known information such as the history of previous returns. In contrast with the historical volatility, the expectation for the next period is calculated using a time-series model that has been selected and estimated using appropriate data.

Stochastic volatility processes are based on an assumption that volatility is *not constant* and therefore it is needed to specify how volatility changes through time. Typical discrete-time models assume that volatility is unobservable and then its stochastic properties may be inferred from absolute or squared returns. Continuous-time models are applied to price options when the assumption of constant volatility is relaxed. A square-root process for volatility allows a rapid calculation of appropriate option prices.

Implied volatility is calculated from an *option price*. It equals the volatility parameter, for which an options market price equals its theoretical price according to a pricing formula. Since the option markets are competitive, the prices must incorporate the markets' expectations about future volatility. Therefore it is reasonable to conjecture that implied volatilities are the best source of information when forecasting volatility.

At any time the values of historical volatility, conditional volatility, stochastic volatility, and implied volatility will typically all be different, due to the fact that different data and assumptions are employed in each calculation.

The volatility of the investment instruments price return is a crucial input factor of almost all pricing, replication, and hedging models (Tichý, 2008). Accurate construction and implementation of this parameter can lead to obtaining more precise information which can help improving the investment decision-making process, therefore help reaching higher gains.

2.8 Financial Modelling and Decision-making

The current phase of economic development can be described by the growth of interconnections among specific activities and processes and increased dynamics of changes. Finance and investment instruments can be regarded as a leading tool to describe and manage all economic activities. Therefore, finance is an important instrument, which allows the regulation, and managing of financial and economic systems and decision-making of particular subjects and institutions. Thus it is obvious that the role of financial modelling as one of the methods can lead to higher quality of financial decision-making processes and knowledge (Zmeškal, et al., 2004).

2.8.1 Financial Models

Financial models cover a complex treatment of the financial problems from the objective of the focus to the application of mathematical tools. Thus there is a wide range of criteria according to which the financial models can be distinguished. These models can be divided into four basic criteria groups:

- Financial Application;
- Decision-making Conditions;

- The Characteristics of Mathematical models;
- Categorization of Stochastic Models.

For the purpose of this thesis, attention will be focused on the decision-making conditions which will be described in the following text.

2.8.2 The Decision-making Conditions

A financial decision-making and modelling can be implemented under various levels of uncertainty of input data and conditions which, in principle, can be broken down to the following areas (Zmeškal, et al., 2004):

- The decision-making under certainty – values and parameters can be expressed by real (crisp) numbers, which is rather an exception in finance.
- The decision-making under uncertainty
  - The decision-making under risk;
  - The decision-making under the vagueness;
  - The decision-making under risk and vagueness.

While in the case of **the decision-making under risk**, the input data can be defined by a probability distribution, in the case of **the decision-making under vagueness** usage of probability distribution is not possible. One way to express the input data is to apply the methodology of fuzzy sets. **The decision-making under risk and vagueness** represents a complex hybrid approach that includes both methods: either some data are defined by a probability distribution and the other by fuzzy sets or all input data are defined by a fuzzy-probability distribution functions.

The focal points of this diploma thesis are alternative methods for parameter estimation of investment instruments financial models which will be performed under certainty and under vagueness. These will be the two levels of input data uncertainties addressed in the following chapters of this diploma thesis.

3 Methods for Estimating the Model Parameters

As the aim of this diploma thesis is the assessment of alternative methods for volatility estimation of investment instruments price models, this chapter of the thesis is focused on the description of four different approaches to volatility estimation. To be able to assess the applicability of those methods, each of the estimated volatilities will be applied to the same financial model which is used for the price evolution estimation. Therefore the parameter estimation, or rather the estimation of volatility, will be evaluated mainly based on the final results of the prediction of the price evolution.

3.1 Preliminary Notions

Prior to the description of approaches to volatility and price estimation, a summary of basic notions important for understanding subsequent sections will be provided. Readers familiar with these notions are free to skip this part.

3.1.1 Numerical Measures to Describe Data

While working with data sets it is essential to be able to describe data numerically. This subchapter will be focused on the explanation of two basic numerical measures. Namely *measures of central tendency and location* and *measures of variability* (Newbold, Carlson, & Thorne, 2013).

Prior to the definition of specific numerical measures it is important to introduce four key terms: *population*, *sample*, *parameter* and *statistic*.

A **population** represents the complete set of all items that interest an investigator. Population size, given by N, can be very large or even infinite. A **sample** is an observed subset of a population with sample size marked as n.

A **parameter** represents a numerical measure that describes a specific characteristic of a population. A **statistic** represents a numerical measure that describes a specific characteristic of a sample.

Measures of Central Tendency and Location

One of the first and basic questions asked by researchers, economists, corporate executives, and anyone with sample data is whether the data in their sample tend to be centred or located around a specific value. To answer this question and reveal the location of the centre of a data set, measures of central tendency and location are used.

The **arithmetic mean** of a set of data is represented by the sum of the data values divided by the number of observations. In the case of the data being the entire population, then the *population mean*, μ, is given by:

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{x_1 + x_2 + \cdots + x_N}{N}, \tag{3.1}$$

where N is a population size.

In the case of data set being in form of a sample, the *sample mean*, $\bar{x}$, is a statistic given by:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \tag{3.2}$$

where n is a sample size.

Percentiles are measures that indicate the location of a value relative to the entire set of data. To find percentiles, data must be arranged in ascending order.

The $\boldsymbol{P^{th}}$ **percentile** represents a value such that approximately P % of the observations are at or below that number. Percentiles separate large data sets into 100ths. The P^{th} percentile can be found as follows:

$$P^{th}\ percentile = value\ located\ in\ the\ \left(\frac{P}{100}\right)(n+1)^{th}\ ordered\ position. \tag{3.3}$$

For the purpose of this diploma thesis the 5^{th} and 95^{th} percentile will be calculated from the selected data sets.

Measures of Variability

The mean alone does not provide a complete or adequate description of data. Another descriptive measures that help describing the data are *measures of variability*. Variation can be spotted in all areas. Even if the mean of two different data sets is the same, usually the measures of variability can distinguish them and describe the sets more accurately.

Two major measures of variability are **variance** and **standard deviation**. The variance represents the average of the sum of squared terms. In case of the *population variance*, σ^2, the mentioned sum is divided by the population size, N:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}. \tag{3.4}$$

In case of the *sample variance*, s^2, the sum is divided by the sample size, n, minus one:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}. \tag{3.5}$$

Because the computing of variance requires squaring the distances, the unit of measurement is in square units. **The standard deviation** restores the data to their original measurement unit, since it is the square root of variance. The standard deviation is used to measure the average spread around the mean.

The population standard deviation, σ , represents the (positive) square root of the population variance:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}}. \tag{3.6}$$

The *sample standard deviation*, s, can be written as:

$$s = \sqrt{s^2} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}. \tag{3.7}$$

In *finance*, the standard deviation of investment instruments returns over time is called **volatility**. Although the common symbol used for volatility is σ, in this diploma thesis the symbol of sample standard deviation $\boldsymbol{s}$ will be kept also for indicating the *volatility*. This is so because all calculations will be performed on sample data sets.

3.1.2 Price and Return

Prior to any calculations it is necessary to compute investment instruments returns from its historical price time series (reasoning discussed in subchapter 2.6). For this purpose it is applicable to use a **logarithmic return**, r, also known as *continuously compounded return*:

$$r = \ln\frac{S_t}{S_{t-1}}, \tag{3.8}$$

where S_t and S_{t-1} are prices of an investment instrument in time t and t minus one.

3.1.3 Stochastic Processes

The price of an investment instrument, S, can be considered as **certain** for a given time period if it is possible to determine its future value, respectively the whole evolution for any

$t \epsilon [0, T]$ with certainty, i.e. with the probability equal to one. Otherwise that quantity has to be considered as **uncertain**.

If it is possible to specify the set of future possible values (discrete or continuous) and at the same time to assign specific probabilities to these values, then the financial quantity S is considered to be **random** or **stochastic**. If those two conditions are not met, this quantity can be described as **vague** and for the purpose of modelling there should be applied an interval approach or the *theory of fuzzy sets*.

Most of the prices of investment instruments are random, i.e. they follow a stochastic process in accordance with appropriate probability distribution. Although it is possible to observe prices only in discrete moments and to watch only discrete changes, for modelling purposes it is generally assumed that it is a continuous change in continuous time. Basic stochastic processes include *Poisson* and *Wiener processes*, *geometric Brownian motion* and *Levy models*.

In this thesis, the stochastic process used for the estimation of price evolution of selected investment instruments will be the geometric Brownian motion (GBM). This price model will be discussed in the subchapter 3.2.1.

3.1.4 Monte Carlo Simulation

Given the task to predict the final price of an investment instrument at selected period of time, it is possible to apply the **Monte Carlo simulation**. This approach is also known as a *stochastic simulation*. The theoretical framework of this subchapter comes from Tichý (2010).

The principles of Monte Carlo simulation are based on the *law of large numbers* – with a growing number of realizations of a random variable the observed characteristics (e.g. mean and variance) and the estimated density function are getting closer to the theoretical assumptions. The realizations of random variable can be obtained both by observing real events and artificially. Whereas in reality it is very difficult to register a high number of observations, by artificial generation this limitation does not apply – it is possible to generate even hundreds of thousands of *random realizations*.

Generally the first step of Monte Carlo simulation is to generate a vector of random variables, ε, in required dimension. Then the corresponding function $f(\varepsilon)$ has to be calculated followed by an evaluation – to determine the required moment of a probability distribution of results, a probability estimation that the given phenomenon will occur, etc.

Pseudorandom Variables

The key step of Monte Carlo simulation is generating the numerical series that will contain independent and identically distributed elements with features of uniform distribution $\mathcal{U}\ [0,1]$. Currently there are many software products that can be used to efficiently generate this numerical series (e.g. *Mathematica*, *Matlab*, *C++*, *R* and *Excel*). In this diploma thesis all calculations will be performed using *Mathematica 9.0*. Because of the practical implementation and the possibility to obtain large-sized numerical series, the algorithms of mentioned software products are usually based on a recursion principle, due to which they are deterministic in nature. For this reason it is not possible to consider the generated sequence as entirely random, thus it is referred to as *pseudorandom*. An alternative procedure represents a QMC technique where the generated sequence is *quasirandom*.

To perform the estimation of the investment instruments price evolution, the Monte Carlo simulation has to be applied to a selected process which is used to model the instrument's price. In this case the GBM. The procedure of this estimation will be discussed in the following subchapter.

3.2 Monte Carlo Simulation with GBM

The Monte Carlo simulation with GBM represents the classic approach to the estimation of the price evolution of investment instruments. The GBM and its application with Monte Carlo simulation will be described in this section.

3.2.1 Geometric Brownian Motion

GBM represents a classic tool for price modelling of an investment instrument since the mid-twentieth century, even though its prime construction was made already at the turn of the century.

There are many types of Brownian motion. One of them is called the exponential Brownian motion (i.e. always positive), also known as the GBM. Most of the attention was given to this very process, due to its specific characteristics. As a general rule investment instruments have a positive value where the return is attributed continuously. Therefore the GBM appears to be an appropriate tool for modelling the price of an investment instrument. The GBM is given by:

$$S_T = S_{0+\tau} = S_0 \cdot exp\left[\left(\bar{x} - \frac{s^2}{2}\right) \cdot \tau + s \cdot \sqrt{\tau} \cdot \varepsilon\right], \tag{3.9}$$

where S represents the price of an investment instrument, $\bar{x}$ and s are sample mean and volatility (sample standard deviation) of the returns of the risky instrument S expressed on a per annum basis, $\tau = T - 0$ is a time period remaining until the end of the observed time period and ε is a random variable from the standard normal distribution, where $\varepsilon \in \mathcal{N}\ [0,1]$.

3.2.2 Direct Monte Carlo Simulation

In the *direct* Monte Carlo simulation, the random, or rather pseudorandom or quasirandom, variables are generated in order to meet the characteristics of the given distribution. The **direct Monte Carlo simulation with GBM** will be described in the following text.

When considered the simplified case, where ω represents a function of one random variable ε, $\omega(\varepsilon)$, and the occurrence of ε can be described by appropriate probability distribution. Provided that a sufficient amount of these random variables ε that comply with the selected probability distribution is generated, the expected value can be simply calculated using *arithmetic mean*.

In this diploma thesis, the direct Monte Carlo simulation with GBM is based on the fact that the dynamics of predicted price of an investment instrument in the time of maturity, S_T, is described using (3.9), where ε is a random variable from the standard normal probability distribution, $\varepsilon \in \mathcal{N}\ [0,1]$. Since the particular state of ω is given by the appropriate combination of pseudorandom variables, in this case ω can be represented by the final price of an investment instrument $\omega = S_T$ or a vector of prices that represents a price evolution during the selected period of time, i.e. discrete sequence $\omega = S_t, S_{t+1}, S_{t+2}, \ldots, S_T$. To obtain the price at the time of maturity, or more precisely, at the end of the selected period, it is necessary to generate the discrete sequence of prices during the selected period $\omega = S_t, S_{t+1}, S_{t+2}, \ldots, S_T$. Individual prices for n^{th} scenario can be obtained by:

$$S_{t+\Delta t}^{(i)} = S_t \exp\left[\Delta S_{\Delta t}^{(i)}\right] = S_t \cdot exp\left[\left(\bar{x} - \frac{s^2}{2}\right) \cdot \Delta t + s \cdot \sqrt{\Delta t} \cdot \varepsilon^{(i)}\right], \qquad (3.10)$$

where the individual parameters have the same meaning as in (3.9) and (i) represents the i^{th} element (price) for n^{th} scenario.

To execute the direct Monte Carlo simulation with GBM it is needed to perform a large number of scenarios (10 000 of scenarios in this thesis), so that the law of large numbers is applicable.

To quantify the estimated final price of an investment instrument, S_T, it is possible to simply calculate an *arithmetic mean* from the estimated final prices computed in each scenario. This can also be performed with prices throughout the predicted time period in order to quantify the estimated price evolution.

The *direct* Monte Carlo simulation is relatively quick and gives us not only the idea of what the price of an investment instrument at the end of selected time period will be, but also the evolution of the price during this period. The down side of this technique is a low convergence. Hence there are other techniques that can improve the estimation by lowering the error of estimation/ improving the convergence, e.g. the technique of antithetic variates Monte Carlo (AVMC), stratified sampling Monte Carlo (SSMC), etc. Additional information regarding these approaches can be found in Tichý (2010).

3.3 Monte Carlo Simulation with GBM with Fuzzy Volatility

The Monte Carlo simulation with GBM with fuzzy volatility will be described in this subchapter. This approach can be understood as an adjusted form of the previous technique. The core of the adjustment lies in the vagueness which represents a new level of input data uncertainty due to the application of fuzzy volatility.

The utilization of fuzzy sets theory in financial models was applied, for example, in (Tichý, & Holčapek, 2011) where the crisp volatility parameter in the standard market model was replaced by a fuzzy random variable which was evaluated by Monte Carlo simulation. Slightly different approach will be introduced in this diploma thesis where the crisp volatility parameter in the GBM will be replace by a fuzzy volatility.

Before describing this approach it is necessary to define fuzzy sets and the parametric representation of LU-fuzzy numbers which represent the basis of this approach.

3.3.1 Fuzzy Sets

The mathematical modelling, or rather financial modelling, has two major obstacles while modelling the real processes. On the one hand it is the *complexity of reality*, due to which it is either impossible to even construct the needed model or the model is so complicated that it is not possible to use it. On the other hand it is the *vagueness* that is caused by the incapability of people to precisely differentiate the reality and precisely define instrumental concepts. Conventional mathematics cannot solve this vagueness. Because of that, during more than three decades, many efforts have arisen to create a mathematical system that would be able to capture vague concepts. Furthermore this kind of mathematical system appeared to be vital for the development of new disciplines such as artificial intelligence.

In 1965 L. A. Zadeh had published a paper that launched a massive development of the *modified fuzzy set theory*. It is a tool which should allow to mathematically describe vague concepts and to work with them. The core term of this theory is **fuzzy set**. The ideology is very simple: if it is not possible to define the exact boundaries of the class determined by the vague term, the decision about whether certain element does or does not belong in that class will be substituted by the measure chosen from some scale. Every element will be assigned with a measure that represents its place and a role in the class. If the scale is arranged, then a smaller measure will express that the given element is closer to the margin of the class. This measure is called the *membership degree* and it represents the grade of membership of an element in a given class. This class, in which every element is characterized by the membership degree, is called the *fuzzy set*. The membership degree represents the degree of our conviction that the given element belongs to a certain fuzzy set. It is important to emphasise that this does not represent the probability. In fact it can be understood as a probability of a fuzzy phenomenon, i.e. an occurrence that is not defined strictly, but by using fuzzy sets. The membership degree can be represented by elements of the interval $[0,1]$ (Novák, 1990).

During the last twenty years, the *fuzzy set theory* has undergone a rapid development both in theory and in application. Nowadays fuzzy sets can be used to describe numerous variables. In the case of this thesis, the modelled parameter will be the *fuzzy volatility*. First, it is necessary to describe **fuzzy numbers** which represent the basis for the construction of fuzzy volatility.

Fuzzy number represents a special fuzzy set in universe of real numbers with which it is possible to model concepts such as “approximately five”, etc. It is possible to perform common arithmetical operations with them, such as addition, subtraction, multiplication and division. Most of the results come from D. Duboise and H. Prade (1978) where it is assumed that the membership degree is set in an interval $[0,1]$. This will be also the premise of this diploma thesis. Further description of fuzzy numbers and their representation continues in the following text.

3.3.2 Fuzzy Numbers and Their LU-representation

Let $\mathbb{R}$ represent the set of real numbers. A **fuzzy number** can be called a mapping $A: \mathbb{R} \to [0,1]$ which is *normal*, i.e. there exists an element $x_0 \in \mathbb{R}$, such that $A(x_0) = 1$, *convex*, i.e.:

$$A(\lambda \cdot x + (1 - \lambda) \cdot y) \geq min(A(x), A(y)), \tag{3.11}$$

for any $x, y \in \mathbb{R}$ and $\lambda \in \langle 0,1 \rangle$. Also which is upper semicontinous and supp(A) is bounded, where $supp(A) = cl\{x \in R \mid A(x) > 0\}$ and cl is the closure operator (Dubois, 1978 and Klir, 1995).

The key characteristic of each fuzzy number is the possibility of their representation by ***α*-*cuts***. The *α-cut* of a fuzzy number *A* is the common set $A_\alpha = \{x \in R \mid A(x) \geq \alpha\}$ and

$$A(x) = \bigvee_{\substack{\alpha \in]0,1] \\ x \in A_\alpha}} \alpha. \tag{3.12}$$

Each *α-cut* $(\alpha \neq 0)$ of a fuzzy number is a bounded interval, i.e.:

$$A_\alpha = [u^- A(\alpha), u^+ A(\alpha)], \tag{3.13}$$

where

$$\begin{aligned} u_A^-(\alpha) &= inf\{x \in \mathbb{R} \mid A(x) \geq \alpha\}, \\ u_A^+(\alpha) &= sup\{x \in \mathbb{R} \mid A(x) \geq \alpha\}. \end{aligned} \tag{3.14}$$

For the simplification of designation of fuzzy numbers discussed in the following text, from this point on the fuzzy number will be represented by u.

According to the *representation theorem* for fuzzy numbers (see Goetschel, 1986) the ***α-cut* setting** can be used to define a fuzzy number. Due to the fact that fuzzy numbers u are upper semicontinous real functions, each *α-cut* can be replaced by its endpoints, i.e. u^- for left endpoint and u^+ for the right endpoint. Thus each fuzzy number can be completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \longrightarrow \mathbb{R}$ (where $u^-(\alpha) = u_\alpha^-$ and $u^+(\alpha) = u_\alpha^+$), satisfying three conditions (Stefanini, et al., 2006):

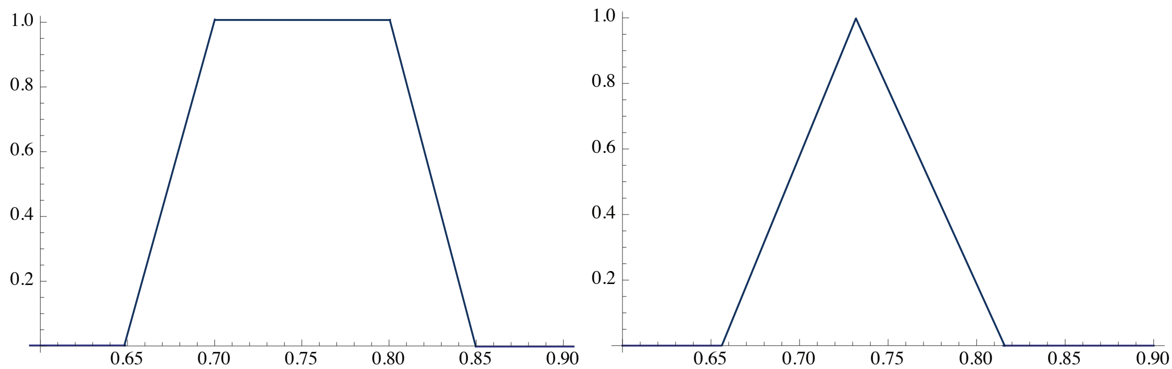
1) u^- is a bounded monotonic non-decreasing function that is left-continuous on]0,1] and the right-continuous for $\alpha = 0$;
2) u^+ is a bounded monotonic non-increasing function which is left-continuous on]0,1] and the right-continuous for $\alpha = 0$;
3) $u_\alpha^- \leq u_\alpha^+$ for any $\alpha \in [0, 1]$.

The fuzzy numbers represented by the pair of functions $u = (u^-, u^+)$ that satisfy conditions 1) – 3) are called **LU-fuzzy numbers**. The abbreviation LU stands for *lower* (u^-) and *upper* (u^+) branch of the fuzzy number u.

The most common models of LU-fuzzy numbers are triangular or trapezoidal. Their attractiveness follows from the simple fuzzy calculus (as addition or multiplication of LU-fuzzy numbers).

A *trapezoidal LU-fuzzy number* is denoted by $u = \langle a, b, c, d \rangle$, where $a, b, c, d \in \mathbb{R}$ and $a \leq b < c \leq d$, and has α-cuts $u_\alpha = [a + \alpha(b - a), d - \alpha(d - c)]$. A *triangular LU-fuzzy number* is denoted by $u = \langle a, b, c \rangle$ and has α-cuts $u_\alpha = [a + \alpha(b - a), c - \alpha(c - b)]$. The graphical example of the trapezoidal and triangular LU-fuzzy number is shown in Figure 3.1. The arithmetic operations for LU-fuzzy numbers are defined in Stefanini et al. (2006).

Figure 3.1 Trapezoidal and Triangular LU-fuzzy Numbers



3.3.3 Parametric Representation of LU-fuzzy Numbers

To model fuzzy volatility, there will be used a more advanced model of LU-fuzzy numbers. It is based on the interpolation of given nodes using rational splines that were proposed by Guerra and Stefanini in (Guerra, & Stefanini, 2005) and developed in (Stefanini, et al., 2006). This model generalises the triangular fuzzy numbers and provides a broad variety of shapes, enabling more precise representation of fuzzy real data, yet the calculus is still very simple.

The **parametric representation of LU-fuzzy numbers** was introduced in Stefanini et al. (2006). It denotes the LU-fuzzy numbers by monotonic curves. The parameterization of fuzzy numbers is obtained by representing the lower and upper branches u_α^- and u_α^+ of u using *cubic rational curves*.

The main interest in parametric representation is maintaining the simplicity of computations by the use of simple local monotonic approximations of the lower and upper branches of LU-fuzzy numbers. These branches will be obtained as *Hermite-type interpolators* (see Tichý, & Holčapek, 2011).

In this thesis the applied function will be the **piecewise rational cubic Hermite parametric function** $P \in C^1[\alpha_0, \alpha_n]$, with parameters $v_i, w_i, i = 0,\ldots,n-1$, that is defined for $\alpha \in [\alpha_i, \alpha_{i+1}]$ and $i = 0,\ldots,n-1$ as:

$$P(\alpha) = P_i(\alpha, v_i, w_i) = \frac{(1-\theta)^3 \cdot u_i + \theta \cdot (1-\theta)^2 \cdot (v_i \cdot u_i + h_i \cdot \delta u_i) + \theta^2 \cdot (1-\theta) \cdot (w_i \cdot u_{i+1} - h_i \cdot \delta u_{i+1}) + \theta^3 \cdot u_{i+1}}{(1-\theta)^3 + v_i \cdot \theta \cdot (1-\theta)^2 + w_i \cdot \theta^2 \cdot (1-\theta) + \theta^3}, \tag{3.15}$$

where u_i and δu_i are, respectively, the real data values and the values of their first derivative values (slopes) at the nodes $\alpha_0 < \cdots < \alpha_N$, $h_i = \alpha_{i+1} - \alpha_i, \theta = (\alpha - \alpha_i)/h_i$ and $v_i, w_i \geq 0$. The parameters v_i and w_i represent the *tension parameters*. The tension parameter used in this thesis leads to a global monotonic cubic rational curve and can be written as (for details see Stefanini, et al., 2006):

$$v_i = w_i = \begin{cases} \frac{\delta u_{i+1} + \delta u_i}{u_{i+1} - u_i}, & for\ u_{i+1} \neq u_i \\ 0, & otherwise \end{cases}. \tag{3.16}$$

The reason for applying this parameter is its derivation from u_i and δu_i. Therefore during any operations with LU-fuzzy numbers (e.g. arithmetic operations or sorting data in ascending or descending order), there is no need to derive separately the values of the tension parameters. Apparently every parametric function $P \in C^1[\alpha_0, \alpha_N]$ can be written in a matrix form as:

$$P = \binom{u}{\delta u} = \begin{pmatrix} u_{\alpha_0} & \cdots & u_{\alpha_N} \\ \delta u_{\alpha_0} & \cdots & \delta u_{\alpha_N} \end{pmatrix}, \tag{3.17}$$

for a partition $\alpha_0 < \alpha_1 < \cdots < \alpha_N$ of the interval $[\alpha_0, \alpha_N]$.

From the definition of cubic rational curves it is obvious that the parametric representation of LU-fuzzy number is continuous and the branches have derivation for any $\alpha = [0,1]$. According to Guerra and Stefanini (2005) it is sufficient to use from three to five nodes to obtain errors of the order of 0.1 %. Therefore the parametric representation represents a good approximation of LU-fuzzy numbers (for more information see Stefanini, et al., 2006).

If the lower and upper branches u_α^- and u_α^+ of u are represented on the trivial decomposition of interval $[0,1]$, with $N = 1$ (without internal points) and $\alpha_0 = 0, \alpha_1 = 1$, then u can be represented by a vector of 8 components:

$$u = (u_0^-, \delta u_0^-, u_0^+, \delta u_0^+; u_1^-, \delta u_1^-, u_1^+, \delta u_1^+), \tag{3.18}$$

where $u_0^-, \delta u_0^-, u_1^-, \delta u_1^-$ represent the lower branch u_α^-, and $u_0^+, \delta u_0^+, u_1^+, \delta u_1^+$ are used for the upper branch u_α^+, by application of a cubic rational curve on the whole interval $\alpha \in [0,1]$.

More generally, a parametric representation of LU-fuzzy number on a decomposition $\alpha_0 < \alpha_1 < \cdots < \alpha_N$ can be written as a system of vectors with eight components:

$$u = (u_{i-1}^-, \delta u_{i-1}^-, u_{i-1}^+, \delta u_{i-1}^+; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=1,\dots,N}. \tag{3.19}$$

where δu_α^- and δu_α^+ denote the slopes associated to the elements u_α^- and u_α^+ respectively.

Because the elements $u_\mathrm{i}^-, \delta u_\mathrm{i}^-, u_\mathrm{i}^+, \delta u_\mathrm{i}^+$ repeat themselves in the system of vectors (for i minus one and i), to simplify this system, it can be rewritten as:

$$u = (u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=1,\dots,N}. \tag{3.20}$$

For the purpose of this diploma thesis the parametric representation of LU-fuzzy number described by (3.20) will be represented by *bimatrix* as:

$$\begin{pmatrix} (u_0^-, \delta u_0^-) & \dots & (u_N^-, \delta u_N^-) \\ (u_0^+, \delta u_0^+) & \dots & (u_N^+, \delta u_N^+) \end{pmatrix}. \tag{3.21}$$

LU-fuzzy Arithmetic Operations

The arithmetic operators associated to the parametric representation of LU-fuzzy numbers can be obtained easily (Stefanini, et al., (2006)).

The *addition* is defined as follows:

$$u + v = (u_i^- + v_i^-, \delta u_i^- + \delta v_i^-, u_i^+ + v_i^+, \delta u_i^+ + \delta v_i^+)_{i=0,1,\dots,N}.$$

The *scalar multiplication* is defined by:

if $k \geq 0$ then $k \cdot u = (k \cdot u_i^-, k \cdot \delta u_i^-, k \cdot u_i^+, k \cdot \delta u_i^+)_{i=0,1,\dots,N}$;

if $k < 0$ then $k \cdot u = (k \cdot u_i^+, k \cdot \delta u_i^+, k \cdot u_i^-, k \cdot \delta u_i^-)_{i=0,1,\dots,N}$.

The *subtraction* is defined as follows:

$$u - v = u + (-v).$$

For fuzzy *multiplication* there will be applied an easy to implement algorithm, based on the application of exact fuzzy multiplication at the nodes of the α-*subdivision*. Define:

$$(u \cdot v)_i^- = min\{u_i^- \cdot v_i^-, u_i^- \cdot v_i^+, u_i^+ \cdot v_i^-, u_i^+ \cdot v_i^+\},$$

$$(u \cdot v)_i^+ = max\{u_i^- \cdot v_i^-, u_i^- \cdot v_i^+, u_i^+ \cdot v_i^-, u_i^+ \cdot v_i^+\},$$

and set the following:

$$y = u \cdot v = (y_i^-, \delta y_i^-, y_i^+, \delta y_i^+)_{i=0,1,\dots,N}.$$

To implement the multiplication the procedure is as follows: let (p_i^-, q_i^-) be the pair associated to the combination of superscripts $+$ and $-$ giving the minimum $(u \cdot v)_i^-$, and similarly let (p_i^+, q_i^+) be the pair associated to the combination of $+$ and $-$ giving the maximum $(u \cdot v)_i^+$, then there are obtained:

$y_i^- = u_i^{p_i^-} \cdot v_i^{q_i^-}$ and $y_i^+ = u_i^{p_i^+} \cdot v_i^{q_i^+}$,

$\delta y_i^- = \delta u_i^{p_i^-} \cdot v_i^{q_i^-} + u_i^{p_i^-} \cdot \delta v_i^{q_i^-}$ and $\delta y_i^+ = \delta u_i^{p_i^+} \cdot v_i^{q_i^+} + u_i^{p_i^+} \cdot \delta v_i^{q_i^+}$,

where the product derivative rule is used to obtain new slopes.

The *exponential function* is defined by:

$$y = Exp\, u = (exp\, u_i^-, exp\, u_i^- \cdot \delta u_i^-, exp\, u_i^+, exp\, u_i^+ \cdot \delta u_i^+)_{i=0,1,\dots,N}.$$

According to the results of experimentation reported in (Guerra, & Stefanini, 2005), the operations above are exact at the nodes α_i and have very small global errors on $[0,1]$. The results have shown that parametric representation models perform well, with a percentage average error of the order of 0.1 %.

To simplify the designation of the *parametric representation model of an LU-fuzzy number* it will be simply referred to as the **fuzzy number** in the rest of this thesis.

3.3.4 Performance of Monte Carlo Simulation with GBM with Fuzzy Volatility

The Monte Carlo simulation with GBM with fuzzy volatility is performed similarly as in the subchapter 3.1.3, therefore the formula is based on (3.10), and can be written as:

$$S_{t+\Delta t}^{(i)} = S_t \exp\,[\Delta S_{\Delta t}^{(i)}] = S_t \cdot exp\left[\left(\bar{x} - \frac{s_{LU}^2}{2}\right) \cdot \Delta t + s_{LU} \cdot \sqrt{\Delta t} \cdot \tilde{\varepsilon}^{(i)}\right], \qquad (3.22)$$

where the individual parameters have the same meaning as in (3.10). The difference is that the volatility is expressed by fuzzy number, and all applied arithmetic operations and functions are created specifically for fuzzy numbers.

The *final estimated price* of an investment instrument, S_T, can be simply calculated as an *arithmetic mean* from the estimated final prices computed in each scenario, using arithmetic operations for fuzzy numbers. This process can be also applied to prices calculated in each moment of selected time interval in order to quantify the estimated price evolution. Naturally the resulting estimated prices of investment instruments are also fuzzy numbers.

3.4 Methods for Estimating the Model Parameters

The aim of this subchapter is to describe four different approaches to the volatility (or the fuzzy volatility) construction. The first approach is the estimation of *crisp volatility* by numerical measures which will be then applied to Monte Carlo simulation with GBM. The remaining three techniques represent fuzzy volatility estimation methods, namely *the box approach*, *the fuzzy partition approach* and *the fuzzy transform approach*. These techniques will be then applied to Monte Carlo simulation with GBM with fuzzy volatility.

3.4.1 Volatility Estimation for Monte Carlo Simulation with GBM

The most simplistic approach of volatility estimation is based on the application of numerical measures by which the crisp volatility is calculated. To utilize this method it is only required to know the historical price data series of an investment instrument from which the revenues are calculate by (3.8). Finally the volatility is simply calculated by (3.7) from the instrument's revenues. This volatility is then applied to the Monte Carlo simulation with GBM described in subchapter 3.2.

The advantage of this approach lies in the fact that the sample data are readily available and the calculation is very simple and fast. However, in the real world it is often difficult to estimate reliable input parameters of stochastic models. This can be caused by a lack of sufficiently long data time series or that the data are too heterogeneous. When the volatility is calculated from a large number of observations, many interesting characteristics of volatility tend to disappear.

3.4.2 Fuzzy Volatility Estimation for Monte Carlo Simulation with GBM with Fuzzy Volatility

An alternative approach to volatility estimation can be found in the application of the fuzzy set theory. Estimating the *fuzzy volatility* can be useful for solving problems that occur with the crisp volatility. By applying the fuzzy volatility to stochastic processes the vagueness of financial reality is incorporated to the model. This enhancement of the estimated input parameter can significantly influence the model's results.

For the fuzzy volatility estimation there will be used fuzzy numbers (i.e. the parametric representation of LU-fuzzy numbers) introduced in subchapter 3.3.3. This fuzzy number has the form of a bimatrix whose elements are pair's $(u, \delta u)$, where u is a value and δu is a slope in u. Therefore, the problem of volatility estimation is reduced to the problem of a specification of pairs $(u, \delta u)$ in such a way that they form a bimatrix. Three different approaches of estimating the fuzzy volatility will be introduced in the following subsections, namely *box*, *fuzzy partition* and *fuzzy transform approach*.

It should be noted that all three approaches to fuzzy volatility estimation will be then used in the Monte Carlo simulation with GBM with fuzzy volatility. As a result there will be three different methods differing by the fuzzy volatility construction approach. All three techniques of fuzzy number construction could be also applied to different parameters in various models.

The Construction of Fuzzy Volatility – Box Approach

The core of the box approach consists in partitioning data into several consecutive boxes of the same length where overlapping of two boxes is permissible. The vector of volatilities computed from data of each box is used for derivation of elements of bimatrix representing fuzzy volatility.

A detailed process of this approach will be presented on a sample of 60 returns of some investment instrument (the case of a five-year historical data series in monthly intervals that will be used in this thesis) which has to be obtained from the instrument's historical prices by (3.8). These 60 historical values must be partitioned into 5 sections (boxes) of 20 with the shift of 10. Therefore in every section, except the first one, first 10 values are repeated from the previous section. The next step is to calculate the sample standard deviation (volatility) from each section by (3.7) and sort these values in ascending order. These values represent the u_i^- and u_i^+ to which the slopes δu_i^- and δu_i^+ are given by:

$$\delta u_i^- = \frac{u_{1+i}^- - u_i^-}{\frac{2}{N-1}} \text{ and } \delta u_i^+ = -\frac{u_{1+i}^+ - u_i^+}{\frac{2}{N-1}}. \tag{3.23}$$

As a result the bimatrix that represents fuzzy volatility is constructed:

$$s_{LU} = \begin{pmatrix} (u_1^-, \delta u_1^-), (u_2^-, \delta u_2^-), (u_3^-, \delta u_3^-) \\ (u_1^+, \delta u_1^+), (u_2^+, \delta u_2^+), (u_3^+, \delta u_3^+) \end{pmatrix}. \tag{3.24}$$

It is necessary to mention that in the case of 36 returns (the case of a three-year historical data series in monthly intervals that will also be used in this thesis) the procedure will be analogical. The data will be partitioned always into 5 parts, but not of 20 but 12 returns with the shift of 6.

The Construction of Fuzzy Volatility - Fuzzy Partition Approach

Assuming in the previous approach that the boxes are mutually disjoint, i.e. each datum belongs precisely into one box, a special case of the technique based on a fuzzy partition of data set is obtained. The vector of volatilities is then computed using the weighted average with respect to fuzzy sets from a fuzzy partition.

The concept of *fuzzy partition* was introduced by Ruspini in (Ruspini, 1969) and then applied due to Perfilieva (Perfilieva, 2006, see also Perfilieva, et al., 2008) in the approximation of functions. The idea of fuzzy partition consists in partitioning data set using several overlapping fuzzy sets, for example triangular shaped functions in such a way that the sum of membership degrees of fuzzy sets forming the fuzzy partition at each datum is equal to one. This condition is called the Ruspini condition (see Ruspini, 1969) and naturally generalises the covering condition in the definition of a classical partition. The vector of volatilities is derived using the weighted average coincides with the application of the *direct discrete fuzzy transformation* (see Perfilieva, 2006).

It should be noted that the *fuzzy transformation* (F-transform) consists of two phases: *direct* and *inverse*. The *direct F-transform* transforms a bounded real function f to a finite vector of real numbers. The *inverse F-transform* sends it back. The outcome is a function $\hat{f}$ that approximates f. The core of the F-transform approach lies in partitioning a given continuous interval using fuzzy sets that are in the theory of F-transform usually called *basic functions*. The (finite) system of basic functions is referred to as *fuzzy partition*. The application of the *inverse F-transform* will be discussed in the next approach.

The key idea of F-transform is a *fuzzy partition of a universe* (an interval of $[a, b]$) into fuzzy subsets. Let $x_1 < \cdots < x_n$ be fixed nodes within $[a, b]$ where $x_1 = a, x_2 = b$ and $n \geq 3$. Fuzzy sets $A_1, \dots, A_n$, that are identified by their membership functions $A_1(x), \dots, A_n(x)$ defined on $[a, b]$, form a *fuzzy partition* of $[a, b]$ if for $k = 1, \dots, n$:

- $A_k: [a, b] \to [0,1], A_k(x_k) = 1$;
- $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$ where for the uniformity of denotation $x_0 = a$ and $x_{\mathrm{n}+1} = b$;
- $A_k(x)$ is continuous;
- $A_k(x), k = 2, \dots, n$, strictly increases on $[x_{k-1}, x_k]$ and $A_k(x), k = 1, \dots, n-1$, strictly decreases on $[x_k, x_{k+1}]$;
- for all $x \in [a, b], \sum_{k=1}^{n} A_k(x) = 1$.

The membership functions $A_1, \dots, A_n$ represent the *basic functions*. The shape of the basic function is not predetermined. It can be chosen additionally. The fuzzy partition is *uniform* if the nodes $x_1, \dots, x_n, n \geq 3$, are equidistant (Perfilieva, 2006).

For the purpose of this thesis, there will be used only the **uniform fuzzy partitions** which are determined using one generating function (see e.g. Holčapek, et al., 2013). The precise definition is as follows.

A **uniform fuzzy partition** is defined using a generating function K which is modified by a parameter h that represents its *bandwidth*. Each basic function of the uniform fuzzy partition is then constructed by an appropriate shift of the modified generating function K.

Let K be a generating function, h and x be a positive real number and $c_i \in \mathbb{R}$. A system of fuzzy sets defined by:

$$A_i(x) = \mathrm{K}\left(\frac{x - c_i}{h}\right), \tag{3.25}$$

for any $i \in \mathbb{Z}$ is called a *generalized uniform fuzzy partition (GUFP) of the real line determined by the triplet* (K, h, c_i) if the Ruspini's condition is satisfied.

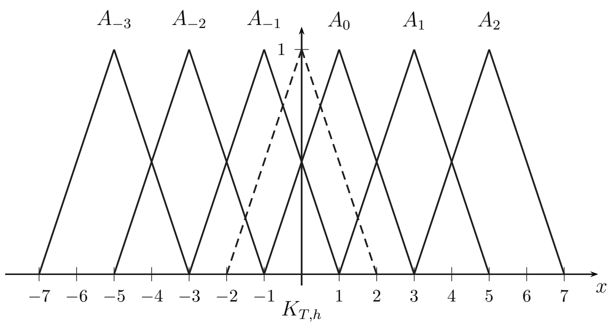
The parameters h and c_i are called a *bandwidth* and *central node*, respectively. The fuzzy sets A_i in (3.25) that form a uniform fuzzy partition of the real line are referred to as *basic functions*. The generating function modified by the bandwidth h is denoted by $K_h(x) = \mathrm{K}\left(\frac{x}{h}\right)$.

In this thesis the applied generating function K will be the **triangle generating function**, which is one of the most useful functions. The K: $\mathbb{R} \to [0,1]$ is defined by:

$$K_T(x) = \max(1 - |x|, 0). \tag{3.26}$$

A part of the uniform fuzzy partition of $\mathbb{R}$ determined by $(K_T, 2, 1)$ is shown in Figure 3.2. For example the basic function A_2 is obtained by transforming K_T to a fuzzy set $K_{T,h}$ with the bandwidth $h = 2$ and shifting the centre $c_0 = 0$ of $K_{T,h}$ to the new centre (node) $c_2 = c_0 + 2h = 1 + 2 \cdot 2 = 5$. The transformed function $K_{T,h}$ is marked on Figure 3.2 using dashed line.

Figure 3.2 Triangle Generating Function



Source: Holčapek, M., et al., 2013, p. 4.

Having defined the triangle generating function and the uniform fuzzy partition determined by this function, now the fuzzy construction process can be presented.

The first step of this method is to determine the partitions from the returns that were calculated by (3.8) from the historical prices. This will be performed similarly to the previous method, thus five partitions will be determined. In the case of 60 returns (a five-year historical data) $h = 15$ and $c_0 = 1$; in case of 36 returns (a three-year historical data) $h = 9$ and $c_0 = 1$.

After determining the parameters for the required partitions, follows the calculation of the mean. Since the components of the direct discrete F-transform are the weighted mean values of the given function where the weights are given by the basic function, for the purpose of this thesis the direct discrete F-transform will be used as the mean calculation. The mean will be calculated in each of five partitions using uniform fuzzy partition with the triangle generating function as (see Perfilieva, 2006):

$$\bar{x}_{c_i} = \frac{\sum_{j=1}^{T} y_{t_j} \cdot \mathrm{K}\left(\frac{t_j - c_i}{h}\right)}{\sum_{j=1}^{T} \mathrm{K}\left(\frac{t_j - c_i}{h}\right)}, \tag{3.27}$$

where t_j represents a time interval j, c_i represents the node, h is the bandwidth and K is the triangle generating function.

The volatility, s_{c_i}, can be then calculated for each partition as a square root of the variance which is given by:

$$s_{c_i}^2 = \frac{\sum_{j=1}^{T} \left(y_{t_j} - \bar{x}_{c_i}\right)^2 \cdot \mathrm{K}\left(\frac{t_j - c_i}{h}\right)}{\sum_{j=1}^{T} \mathrm{K}\left(\frac{t_j - c_i}{h}\right)}, \tag{3.28}$$

where the individual parameters have the same meaning as in (3.27).

Having established the partial volatilities s_{c_i}, it is necessary to sort them in ascending order. Then the slopes are assigned to each partial volatility by (3.23). As a result the fuzzy volatility is constructed and can be written as a bimatrix, see (3.24).

The *premise* of this method is that by using fuzzy partitions the historical data can be defined more precisely, and therefore the complex reality can be described more accurately. The aim of applying this technique is to assess whether this premise will be confirmed by the obtained results.

The Construction of Fuzzy Volatility – Fuzzy Transform Approach

In the previous two approaches, the volatility was derived directly from boxes or a fuzzy partition of data set. This subchapter is dedicated to the most advanced technique for the fuzzy volatility estimation whose core consists in a transformation of data set into fuzzy numbers. The fuzzy volatility is then derived using usual statistical tools applied to fuzzy numbers. The transformation from data into fuzzy data is given using both *discrete direct* and *inverse F-transform* which were introduced in previous subchapter.

The first step of this approach is to perform the direct discrete F-transform of the original historical data. This is performed by (3.27) described in previous subchapter. To obtain the approximated data series the inverse discrete F-transform has to be executed.

The **inverse discrete F-transform** is given by the inversion formula. It approximates the original function in such a way that a universal convergence can be established.

Let function f be given at nodes $x_1, \dots, x_j \epsilon [a, b]$ and $\boldsymbol{F}_n[f] = [F_1, \dots, F_n]$ be the direct discrete F-transform of f with respect to $A_1, \dots, A_n$. Then the *inverse discrete F-transform* function is given by:

$$f_{F,n}(x_j) = \sum_{k=1}^{n} F_k \cdot \mathrm{A}_k(x_j). \tag{3.29}$$

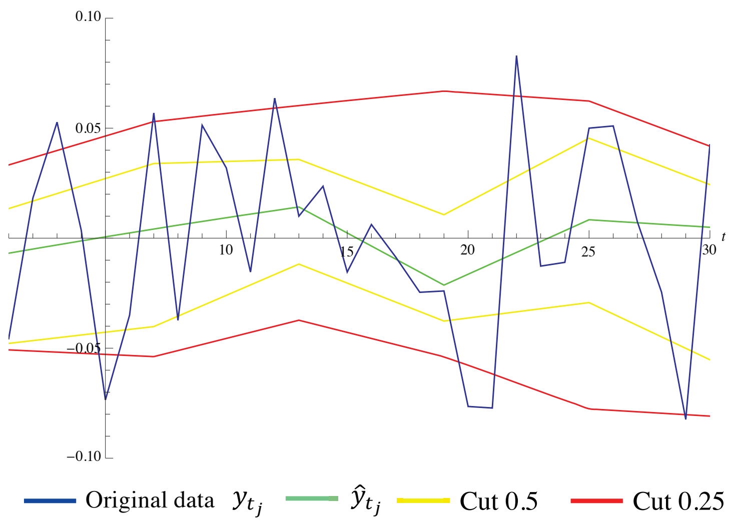
Consequently in the case of this thesis the inverse discrete F-transform of the original historical data into *approximated data series* is given by:

$$\hat{y}_{t_j} = \sum_{i=1}^{n} \bar{x}_{c_i} \cdot \mathrm{K}\left(\frac{t_j - c_i}{h}\right), \tag{3.30}$$

where the individual parameters have the same meaning as in (3.27).

Having obtained an approximated data series, the next step is to create (in the case of this thesis) five **cuts** throughout the entire time series t. For better reference, these cuts are depicted in Figure 3.3.

Figure 3.3 Cuts Through Data Time Series



The first two cuts will be referred to as upper and lower cut 0.5 (yellow line). The second two cuts will be referred to as upper and lower cut 0.25 (red line). As result four new approximated data time series will be generated, two above and two below the approximated data series $\hat{y}_{t_j}$. The calculation of these cuts is performed similarly for upper and lower cuts.

First, the new data time series have to be generated. These series are created from all original data y_{t_j} that lie above, or under the approximated series $\hat{y}_{t_j}$ (or the resulting approximated cut 0.5 in the case of cut 0.25). If a value in specific t_j is not located above, or as the case may be, below the required data series, it is substituted by a value from the boundary time series.

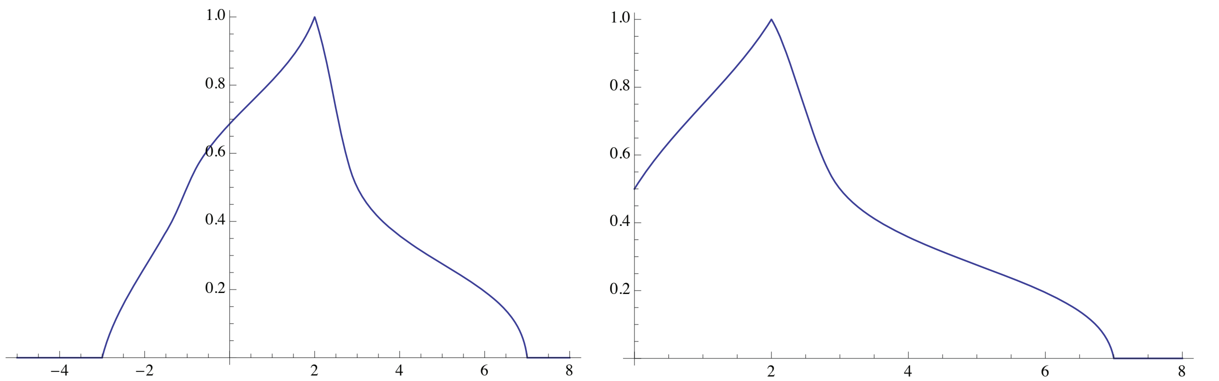
After determining data in each cut, the next step is to perform the discrete direct and inverse F-transform on these data series in every cut by (3.27) and (3.30). Having determined all of the approximated data in every cut, the next step is to group all five numbers (one from each cut) in every time interval t_j together. In this way, sets of five numbers will be obtained to which it is only needed to assign the slopes by (3.23) to obtain a **fuzzy historical data time series**.

This *fuzzy series* represents the source fuzzy data for calculating the **fuzzy volatility**. For this purpose it is necessary to compute the fuzzy mean that is given by (3.2). The final step is the calculation of fuzzy volatility by (3.7) that can be written as a bimatrix as shown in (3.24). Both calculations are performed using arithmetic operations for fuzzy numbers.

While determining fuzzy volatility from fuzzy data, a problem may occur in the calculation of the square root of the fuzzy variance. Such situation may arise where a part of the fuzzy variance has negative values. Although it is a square product, therefore it should not reach negative values, in the case of fuzzy numbers this situation may occur. It is caused by the fact that if fuzzy number represents a value near zero it expresses “approximately zero” and therefore the negative values can occur. For this reason the square root cannot be computed. To solve this problem it is possible to formulate just the positive part of that fuzzy number where zero replaces nodes with negative values, see Figure 3.4. This can be formulated in Mathematica software by:

$$\boldsymbol{PositivePartFn}[\boldsymbol{A\_}] := \boldsymbol{Table}[\boldsymbol{If}[\boldsymbol{A}[[\boldsymbol{t},\boldsymbol{j},\mathbf{1}]] < \mathbf{0}, \{\mathbf{0}, \boldsymbol{A}[[\boldsymbol{t},\boldsymbol{j},\mathbf{2}]]\}, \boldsymbol{A}[[\boldsymbol{t},\boldsymbol{j}]]], \{\boldsymbol{t},\mathbf{1},\mathbf{2}\}, \{\boldsymbol{j},\mathbf{1},\boldsymbol{Length}[\boldsymbol{A}[[\mathbf{1}]]]\}]. \tag{3.31}$$

Figure 3.4 Fuzzy Number and Positive Part of Fuzzy Number



4 Application of Selected Methods and Their Evaluation

In this chapter methods described above will be applied to three different investment instruments and the calculations will be based on two different ranges of historical data time series. This chapter will be divided into two main parts.

The *first part* will be dedicated to the application of methods on the unit of the Pioneer Investments equity mutual fund. Additionally, the calculations will be based on two different ranges of historical data time series to assess its impact on the estimation. Afterwards all methods and obtained results will be evaluated.

In the *second part* of this chapter the studied methods will be implemented on two different investment instruments, considering just one range of historical data time series. Finally all observed results will be evaluated.

4.1 Input Data of Selected Investment Instruments

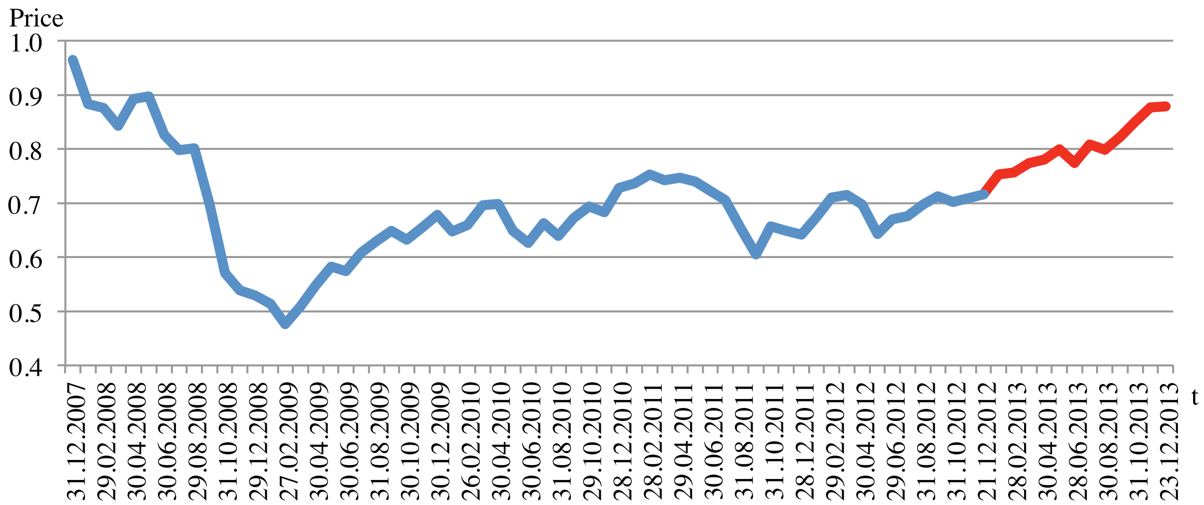
The discussed methods will be applied to three types of investment instruments. These instruments are:

- The unit from Pioneer Investments equity fund;
- Foreign exchange rate (CZK/EUR);
- Gold.

Pioneer Investments is a global investment group that offers products and long-term solutions with the aim of protection and appreciation of the client's investments since 1928. In the Czech Republic Pioneer Investments provide services since 1995 for both individual and institutional clients and foundations. Pioneer represents an open-end mutual fund.

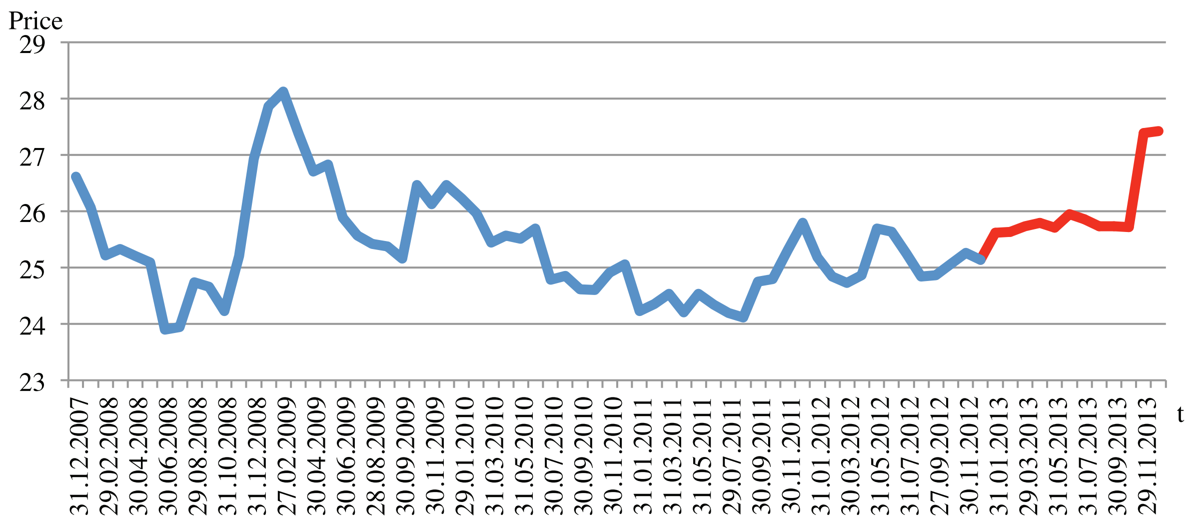
The Pioneer Investments **equity fund** generates appreciation by investing in equity and equity-related securities in the global market. This fund seeks investments into companies with strong fundamental characteristics across a variety of different sectors and industries. The equity fund secures investments in foreign currency against the currency risk via foreign exchange market instruments. Prices regarding the unit from equity fund were obtained from the *Pioneer Investments official webpage* (see Annex no. 1 and 4). For the evolution of the units price see Figure 4.1, where the red line denotes prices in the time period to which the estimations will be performed.

Figure 4.1 The Evolution of the Equity Units Price from December 2007 to December 2013



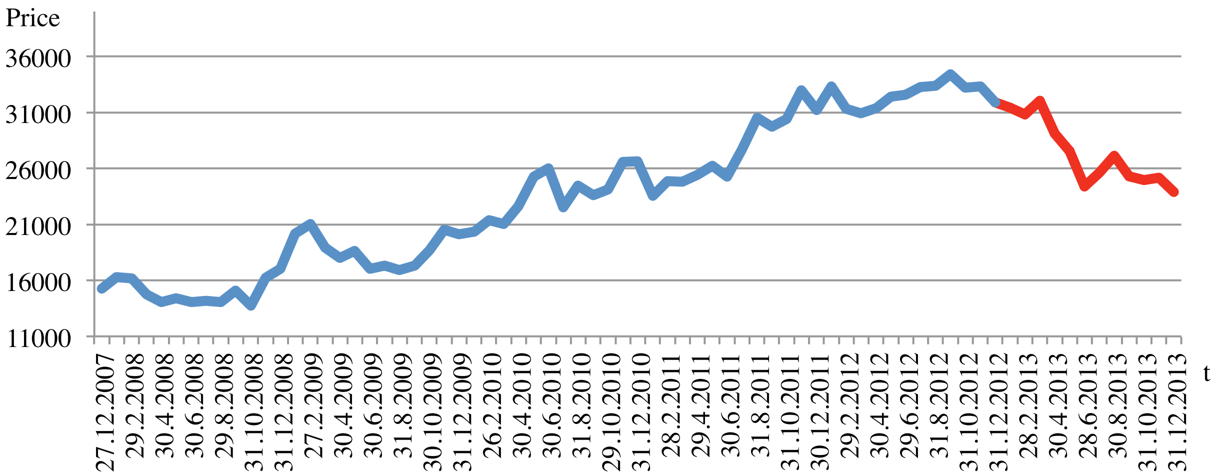
The **foreign exchange (FX) rate of Czech koruna (CZK) to euro (EUR)** has been chosen as an FX rate representative. It is necessary to mention that on the 7th of November 2013 the Czech National Bank (CNB) has begun intervening to weaken the CZK in order to avoid deflation. This intervention has resulted in distinct depreciation from 25.785 EUR/CZK on the 6th of November 2013 to 26.850 EUR/CZK on the 7th of November 2013. At the end of the year 2013, the FX rate reached 27.425 EUR/CZK. The aim of the CZK/EUR FX rate estimation will be the assessment whether the results of performed estimations will incorporate the distinct depreciation of the FX rate. The rates were obtained from the *CNB's official webpage* (see Annex no. 2 and 4). Their evolution from December 2007 to December 2013 is shown in Figure 4.2 where the red line indicates prices in the time period to which the estimations will be performed.

Figure 4.2 The Evolution of the CZK/EUR FX rate from December 2007 to December 2013



Gold is the representative of real assets. This investment instrument is mainly bought as a protection against devaluation of savings. Since the price of gold has vastly risen from ca. 2002 to 2011, and since then the price has started to decrease, it will be interesting to see how the selected methods will project the future price evolution based on the historical data. The prices for one troy ounce in CZK were obtained from the webpage *kurzy.cz* (see Annex no. 3 and 4). The evolution of the gold price from December 2007 to December 2013 is shown in Figure 4.3. The red line denotes prices in the time period to which the estimations will be calculated.

Figure 4.3 The Evolution of the Gold Price from December 2007 to December 2013



The estimation of the instruments' price evolution will be performed for a time period of one year, that is from January to December 2013. This time frame was chosen with the intention of improving the understanding of the outcome of selected methods. Since the real prices in 2013 are already known, it is possible to compare the real price evolution with the estimation, and get a better understanding of the estimation accuracy.

The input data will be represented by monthly prices of the instruments in CZK. As it was mentioned earlier, two historical price time series will be taken under consideration. The five-year time series for the period of December 2007 to December 2012, and the three-year time series for the period of December 2009 to December 2012. To obtain more accurate results the monthly data have not been expressed as average values, but as the values at the end of each month. Thereby the random fluctuations are taken into account to a greater extent.

Prior to any estimation, the returns have to be computed from instruments prices by (3.8). Further calculations will proceed from these returns.

Since the studied methods will be applied to the same instruments, they also have the same input data which are presented in Table 1 differentiated by the investment instrument and the source data range. It is important to mention that all numerical measures calculated in this thesis will be expressed on a per annum basis (p.a.).

Table 1 Input Data Differentiated by the Investment Instrument and the Source Data Range

	Symbol	Equation	5 years	3 years		
			Equity fund	Equity fund	Gold	CZK
Initial price	S_0	-	0.7164	0.7164	31 890.45	25.14
Mean (p.a.)	$\bar{x}$	(3.2)	-0.0595	0.0182	0.1534	- 0.0171
Volatility (p.a.)	s	(3.7)	0.1878	0.1456	0.2005	0.0521

Keeping in mind that all price evolution estimations will be performed using *Monte Carlo simulation with GBM* for the time period of one year, and the historical price time series are monthly values, the simulation for each realization will be performed in the time interval of one month, i.e. $\Delta t = \frac{1}{12}$. The Monte Carlo simulation is therefore calculated for $N = 12$ steps (intervals), thus the length of one step is $\Delta t = \frac{1}{12}$. The simulations will be performed for $n = 10\ 000$ realizations. For better reference, these input parameters are shown in Table 2.

Table 2 Input Parameters for Monte Carlo Simulation with GBM, or with Fuzzy Volatility

Parameter	Symbol	Value
Number of steps	N	12
Time interval	Δt	1/12
Number of realizations	n	10 000

In the next subchapter the studied methods will be applied to the equity unit of Pioneer Investments mutual fund. The price estimation will be calculated from three- and five-year range of the historical price time series.

4.2 Monte Carlo Simulation with GBM

The first applied approach of the volatility estimation, and by extension the estimation of price evolution, is the volatility calculated by (3.7) integrated in the Monte Carlo simulation with GBM (M1).

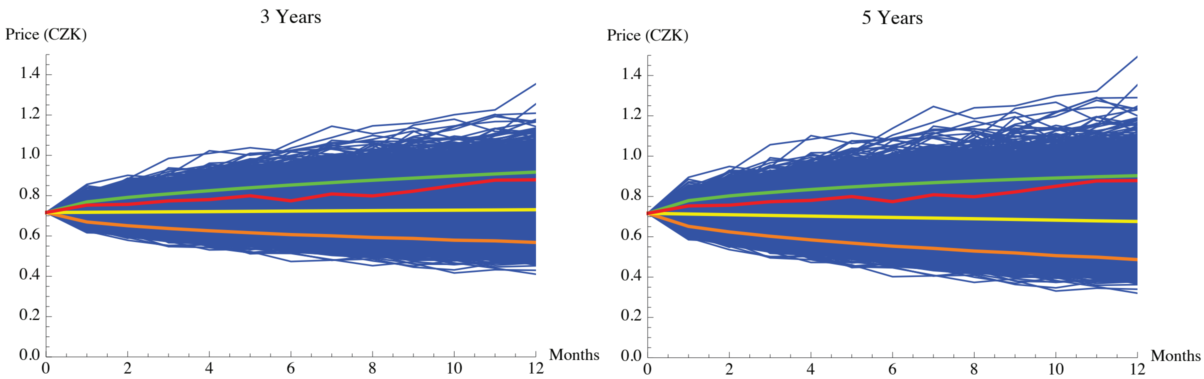
The Estimation of Price Evolution

The price evolution of equity unit is obtained by substituting the input data presented in Table 1 and Table 2 into (3.10) and performing this calculation in 10 000 realizations.

The resultant evolution of the equity unit's price from each of n realizations is shown in Figure 4.4. In addition the figure contains the mean (yellow line), the 5^{th} (orange line) and the 95^{th} (green line) percentile throughout the estimated price evolution of the equity unit. For a better understanding of the performed estimation, this figure also contains the known real evolution of the price (red line).

The real price evolution is in both cases above the estimated average. Even though the scenarios of price evolution obtained from the five-year historical data have a wider price span, more accurate estimation was obtained using the three-year price series.

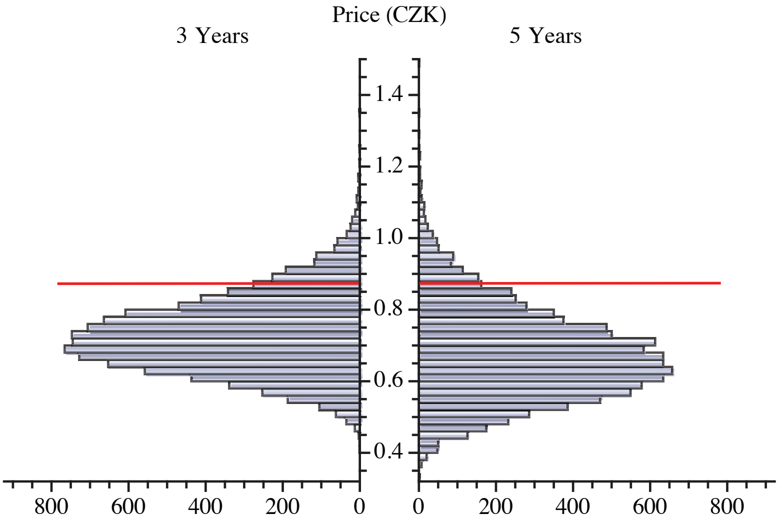
Figure 4.4 Representation of 10 000 Realizations of Monte Carlo Simulation with GBM



The Estimation of the Final Price

The probability distribution of the estimated final price is shown in Figure 4.5. The final price of the equity unit can be simply obtained by calculating an arithmetic mean from the estimated prices in the 12^{th} time interval (December 2013). Using the three-year historical data series the final price of the equity unit is 0.7303 CZK. The estimated final price calculated from the five-year price series is 0.6759 CZK. Since the initial price of the equity unit as at December 2012 was 0.7164 CZK, based on the three-year price series it is estimated that the price of the unit will increase, whereas based on the five-year price series it is estimated that the price will decrease. It is obvious that the determination of the source data range is crucial in the price estimation process.

Figure 4.5 Probability Distribution of the Equity Units Final Price and the Real Price



The real price of the equity unit has increased since December 2012, and has reached 0.8788 CZK in December 2013. It is therefore obvious that the application of the shorter historical price series results into more precise estimation.

4.3 Monte Carlo Simulation with GBM with Fuzzy Volatility

This subchapter is dedicated to a more complex representation of volatility using fuzzy numbers. As it was mentioned in the previous chapter, this thesis is focused on three different approaches for fuzzy volatility estimation. These methods will be applied to the Monte Carlo simulation with GBM with fuzzy volatility which will be constructed using *the box approach (M2)*, *the fuzzy partition approach (M3)* and *the fuzzy transform approach (M4)*.

These methods will be used to show greater possibilities of the volatility construction using fuzzy numbers and their resulting impact on the estimation of the price evolution.

4.3.1 Fuzzy Volatility Constructed Using Box Approach

In this subchapter, the Monte Carlo Simulation with GBM with fuzzy volatility constructed by the box approach, i.e. the M2, will be applied.

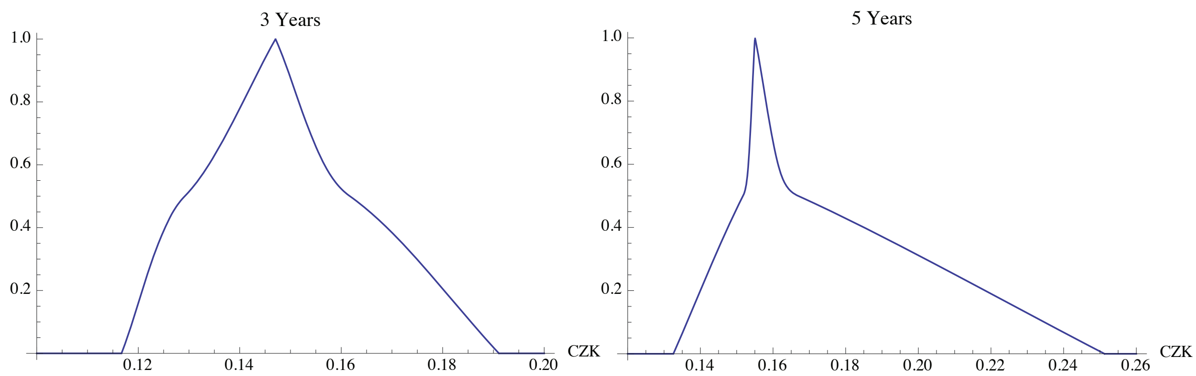
The first step of the M2 is to construct the fuzzy volatility according to the procedure explained in subchapter 3.4.2. The instruments' returns are partitioned into 5 parts of 20, or 12, with the shift of 10, or 6 (depending on the historical data time series). Afterwards the volatility is calculated by (3.7) from each partition and slopes are assigned to each volatility by (3.23). Finally the constructed fuzzy volatility can be represented by bimatrix as:

$$s_{LU}^{3} = \begin{pmatrix}(0.1167, & 0.0249), (0.1292, & 0.0607), (0.1471, & 0.0358)\\ (0.1911, & -0.0588), (0.1617, & -0.0881), (0.1471, & -0.0293)\end{pmatrix}, \tag{4.1}$$

$$s_{LU}^{5} = \begin{pmatrix}(0.1326, & 0.0384), (0.1518, & 0.0449), (0.1550, & 0.0065)\\ (0.2513, & -0.1688), (0.1669, & -0.1925), (0.1550, & -0.0238)\end{pmatrix}. \tag{4.2}$$

The constructed fuzzy volatilities of the equity unit returns are shown in the Figure 4.6, differed by historical price time series. On the one hand the fuzzy volatility computed from the five-year historical data has a wider base, thus with lower membership degree the volatility of returns has a wider span of values than in the other approach. On the other hand the top of this fuzzy volatility is narrower, therefore the values of the fuzzy volatility estimated with higher membership degree have narrower range. It can be seen that it is also tilted to the left. This is caused by the fact that the majority of partial volatilities had lower values, and only one partial volatility reached higher value. This can be attributed to the significant decrease of the unit's price in 2008 (see Figure 4.1). Since the volatility of the historical returns in the three-year time period was more balanced, the fuzzy number is not heavily tilted.

Figure 4.6 The Fuzzy Volatility



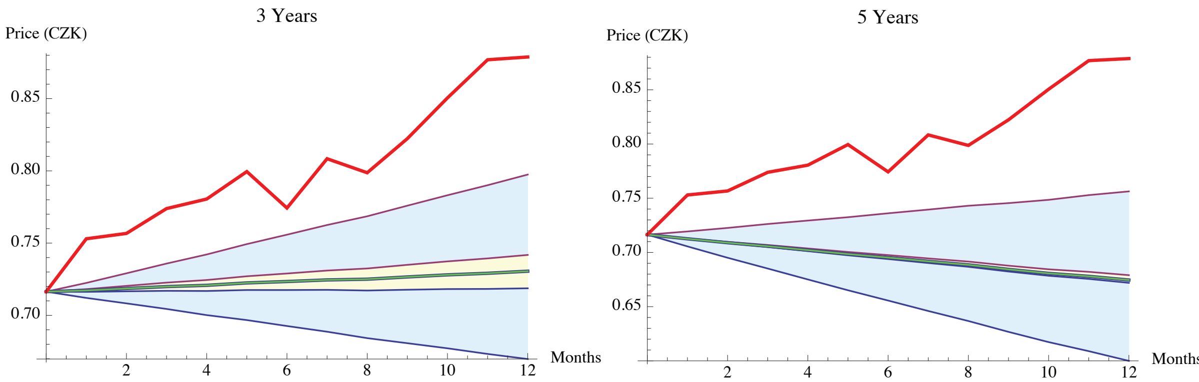
In this approach the fuzzy volatility contains much more information than the conventional volatility represented by a crisp number. While many interesting characteristics of the historical return series were not included in the crisp volatility, by using the fuzzy numbers it was possible to incorporate these characteristics into the fuzzy volatility, and therefore project the complex financial reality into the financial models more accurately.

The Estimation of Price Evolution

After the fuzzy volatility is obtained it is possible to predict the equity units price evolution in 2013 by (3.22), i.e. the Monte Carlo simulation with GBM with fuzzy volatility (M2). The input data are shown in Table 1, Table 2, (4.1) and (4.2).

The next step, after obtaining the estimated price evolution, is the calculation of the estimated average price evolution. It can be performed by a simple arithmetic mean using fuzzy arithmetic operations. The estimated average price evolution and the real price evolution of the equity unit are shown in Figure 4.7.

Figure 4.7 Estimated Average Price Evolution Compared with the Real Price Evolution



This figure shows three different α-cuts through the predicted one-year period. The α-cut 1 (green line) represents the "peak" of the fuzzy number, i.e. a value that has a membership degree of one. This value can be perceived as some kind of a "mean" of a fuzzy number. But still it is important to remember that the fuzzy number is not just one α-cut, but it contains a broader amount of information and should be assessed as a whole. The α-cut 0.8 (the yellow surface) is bounded by two lines which represent the limit values of that α-cut. For the purpose of correct interpretation of the fuzzy price evolution, the top line can be perceived as the possible optimistic evolution of the unit's price with the membership degree of 0.8 which should be taken into account by risk-loving investors. On the other hand, the bottom line should be determinant for risk-averse investors. The interpretation of the price evolution can change, given the additional information about the specific investment instrument from e.g. fundamental analysis. If the investor had an information about the perceptivity of a specific industry or company, he could choose which α-cut and which bound is more determinative. The same interpretation approach can be used on the α-cut 0.2 (blue surface).

Neither the three-year nor the five-year historical price series ensured that the real evolution of equity unit's price would be included in one of the fuzzy numbers. On the other hand, the three-year historical price series appears to be more appropriate for one-year price estimation, while it has the same increasing trend as the real price evolution.

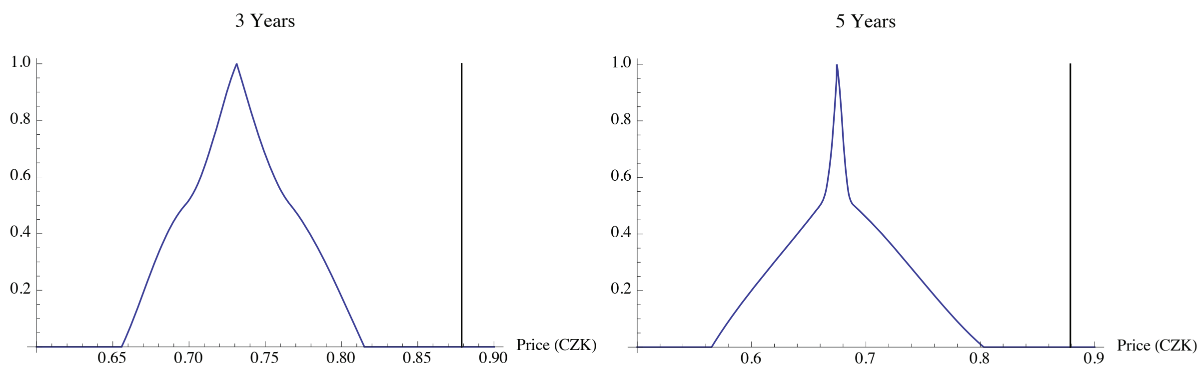
The Estimation of the Final Price

The estimated final price as at December 2013, which was obtained by calculating a simple arithmetic mean (using fuzzy arithmetic operations) from the final prices computed in each scenario, is depicted in Figure 4.8.

It can be seen that the shapes of fuzzy prices are similar to the applied fuzzy volatilities, but they are both almost entirely symmetrical. The estimated fuzzy price calculated from five-year data series has a wider span in the lower α-cuts. By applying the fuzzy volatility with wider base, the resulting fuzzy price incorporates a higher amount of price values.

The α-cut 1 of the estimated average final price is 0.7311 CZK (calculated from three-year historical data) and 0.6746 CZK (calculated from five-year historical data). While the initial price was 0.7164 CZK, it is obvious that the estimated final price based on the three-year historical data has increased, whereas the estimated price based on five-year historical data has decreased. Since the real value of the unit's price has increased, it is evident that the estimated price calculated from the three-year historical price series is closer to the real price. This approach appears to lead to more accurate estimation.

Figure 4.8 The Estimated Average Final Price and the Real Price of the Equity Unit



4.3.2 Fuzzy Volatility Constructed Using Fuzzy Partition Approach

In this section there will be applied the Monte Carlo simulation with GBM with the fuzzy volatility constructed by the fuzzy partition approach (M3).

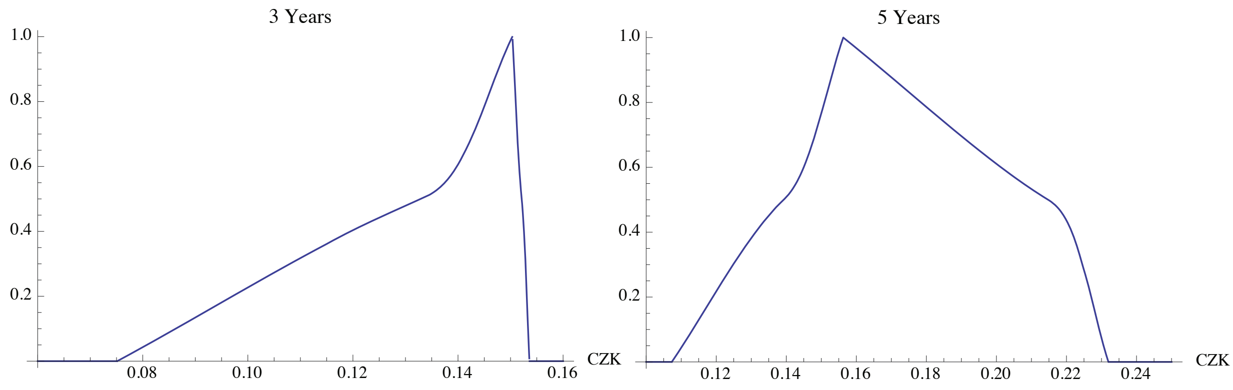
The fuzzy partition approach is based on the uniform fuzzy partitions with the triangle generating function. The procedure of this technique follows the steps discussed in subchapter 3.4.2. The fuzzy volatility is calculated by the square root of (3.28), substituting the input data from Table 1. This way a fuzzy volatility is constructed, and on the basis of both historical price series is given by:

$$s_{LU}^{3} = \begin{pmatrix} (0.0751, \quad 0.1157), (0.1330, \quad 0.1505), (0.1504, \quad 0.0348) \\ (0.1536, -0.0030), (0.1521, -0.0065), (0.1504, -0.0035) \end{pmatrix}, \tag{4.3}$$

$$s_{LU}^{5} = \begin{pmatrix} (0.1073, \quad 0.0639), (0.1393, \quad 0.0979), (0.1563, \quad 0.0341) \\ (0.2320, -0.0345), (0.2148, -0.1514), (0.1563, -0.1170) \end{pmatrix}. \tag{4.4}$$

The constructed fuzzy volatilities of the equity unit returns are depicted in Figure 4.9. The fuzzy volatility calculated from the five-year historical return series is much higher than the fuzzy volatility based on the three-year return series. The higher values of that fuzzy volatility are associated with the distinct change of the unit's price in 2008 (see Figure 4.1). Since the higher volatility was captured by two partial volatilities, the resultant fuzzy volatility is tilted slightly to the left. In the case of the fuzzy volatility calculated on the basis of the three-year data series, the tilt to the right can be explained by one low partial volatility value which was calculated mainly from the returns in 2012.

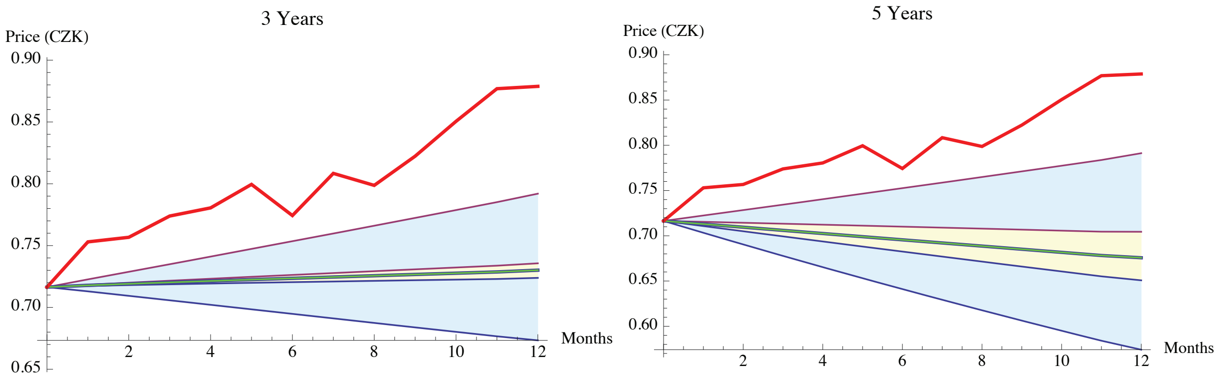
Figure 4.9 The Fuzzy Volatility



The Estimation of Price Evolution

When the fuzzy volatility is constructed, the estimation of equity unit's price evolution in 2013 can be performed by (3.22). The input data are given in Table 1, Table 2, (4.3) and (4.4). The next step is the calculation of the arithmetic mean from the resulting estimated price evolution using fuzzy arithmetic operations. The estimated average final price evolution and the real price evolution of the equity unit is shown in Figure 4.10.

Figure 4.10 Estimated Average Price Evolution Compared with the Real Price Evolution



The interpretation of the α-cuts discussed in previous subchapter applies also to this price evolution. The results of the estimations are similar as in the previous method. Using both historical data series, the evolution of real price has not been predicted in any month. Still the estimated price evolution calculated from the three-year historical data series is closer to the real price evolution.

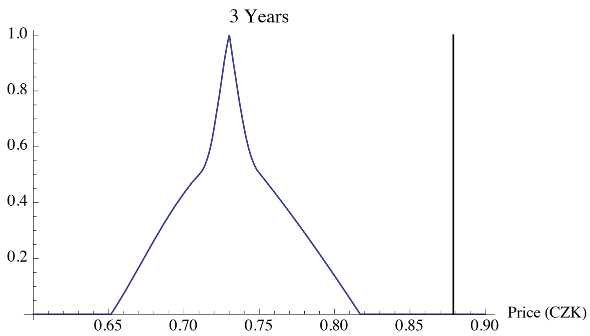
The Estimation of the Final Price

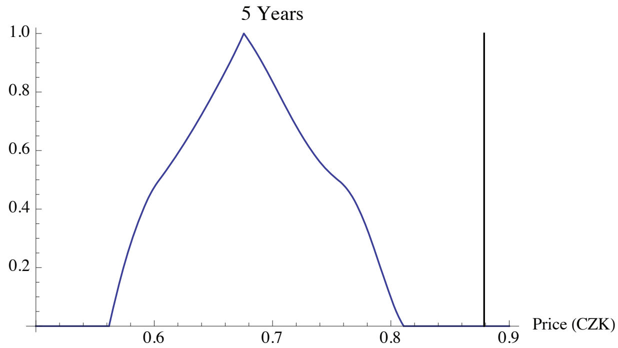
The estimated average final price and the real price of the equity unit as at December 2013 are shown in Figure 4.11.

Even though the estimated final fuzzy prices are almost symmetrical, a similarity can be observed between them and the associated fuzzy volatilities. The resemblances lie mainly in the relative width of the lower and higher α-cuts.

The α-cut 1 of the average final price is 0.7301 CZK (three-year data series) and 0.6760 CZK (five-year data series). Since the initial price of the equity unit as at December 2012 was 0.7164 CZK, it is obvious that the estimated final price based on the three-year historical data has increased, while based on the five-year historical data the price has decreased. While the real price of the equity unit as at December 2013 reached 0.8788 CZK, it appears that by using the shorter historical data series the obtained estimation is more accurate.

Figure 4.11 The Estimated Average Final Price and the Real Price of the Equity Unit





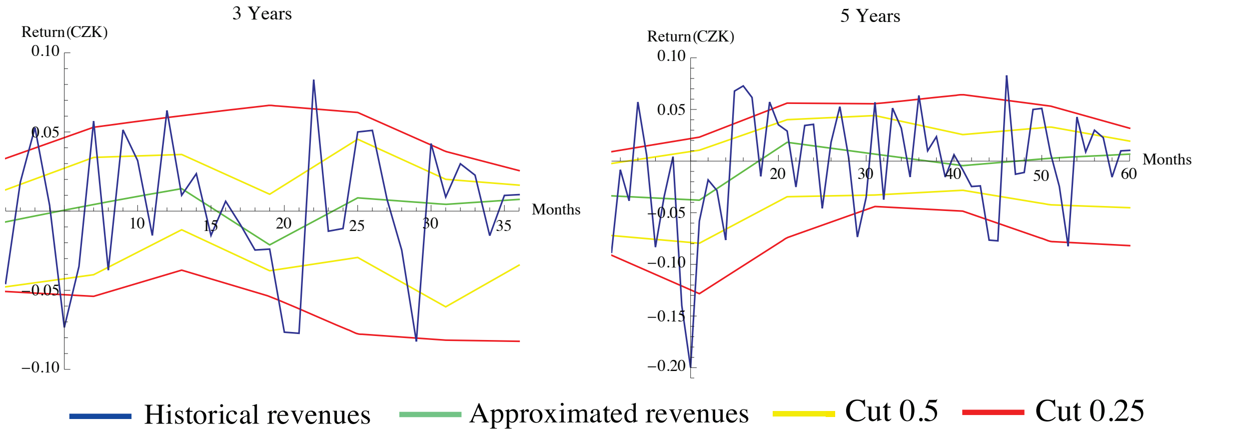
4.3.3 Fuzzy Volatility Calculated Using Fuzzy Transform Approach

In this subchapter there will be applied the Monte Carlo simulation with GBM with fuzzy volatility which will be constructed using the fuzzy transform approach (M4).

The last and the most advanced technique to construct the fuzzy volatility is based on the discrete direct and inverse F-transform where all source data will be transformed into fuzzy numbers.

The first step of this approach is the transformation of the historical returns time series into fuzzy data. This step is done by the following procedure. First the original historical returns time series is transformed by the direct discrete F-transform using the calculation of mean by (3.27). The next step is to calculate the inverse discrete F-transform by (3.30). Having generated the approximated returns time series, four cuts of the original data can be constructed according to the procedure described in subchapter 3.4.2. Next the discrete direct and inverse F-transform is performed on the new data series in each cut by (3.27) and (3.30). The approximated cuts of the historical returns time series are shown in Figure 4.12.

Figure 4.12 The Historical Returns Time Series with Five Approximated Cuts

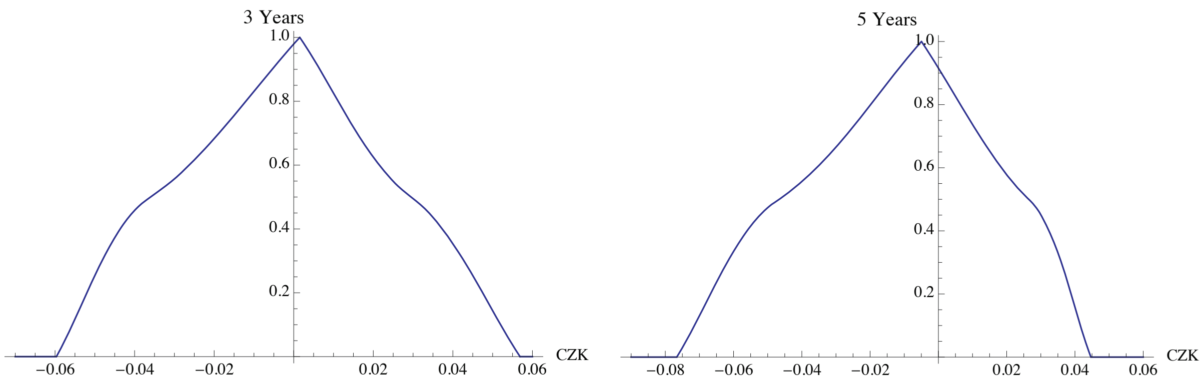


The data in approximated cuts in each time interval represent the values u of a fuzzy number. The last step to obtain the fuzzy data is to assign the slopes by (3.23) to the five value sets in each time interval.

The Fuzzy Volatility Construction Process

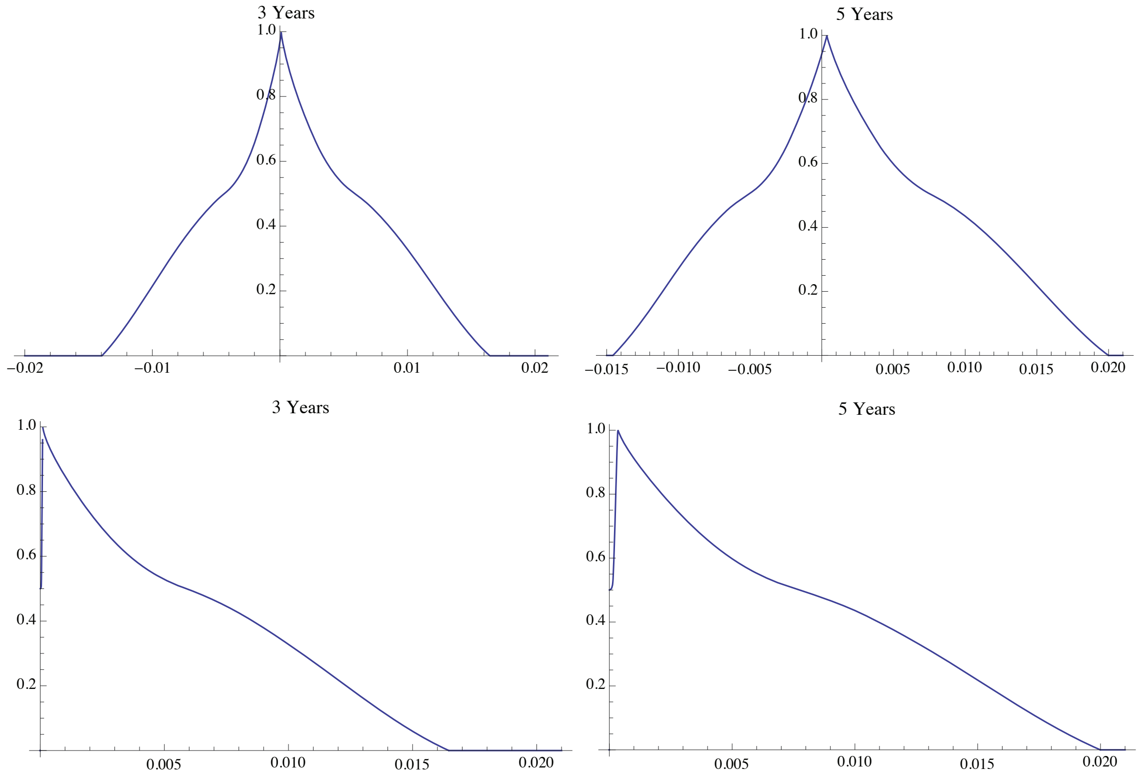
To construct the fuzzy volatility from fuzzy data it is necessary to quantify the fuzzy mean value. This value is computed by (3.2) using fuzzy arithmetic operations. The resulting fuzzy mean, distinguished by the applied historical fuzzy returns time series, is depicted in Figure 4.13. It can be seen that the fuzzy mean calculated from the five-year data series is less symmetrical than the other fuzzy mean and is located slightly more in the negative values. This indicates the decreasing trend of the unit's price.

Figure 4.13 The Fuzzy Mean of the Equity Unit



The final step is the calculation of fuzzy volatility by (3.7) using fuzzy arithmetic operations. As it was mentioned earlier, the fuzzy volatility represents a square root of the fuzzy variance. Therefore it is necessary that the fuzzy variance has positive values. Even though variance is a square product, therefore it should not reach negative values, in the case of fuzzy numbers this situation can occur (reasoning discussed in subchapter 3.4.2). The solution lies in applying just the *positive part of the fuzzy number* by (3.31) in further calculations. The original fuzzy variance and the fuzzy variance created with just the positive part of that fuzzy number is shown in Figure 4.14.

Figure 4.14 The Fuzzy Variance and The Positive Part of the Fuzzy Variance



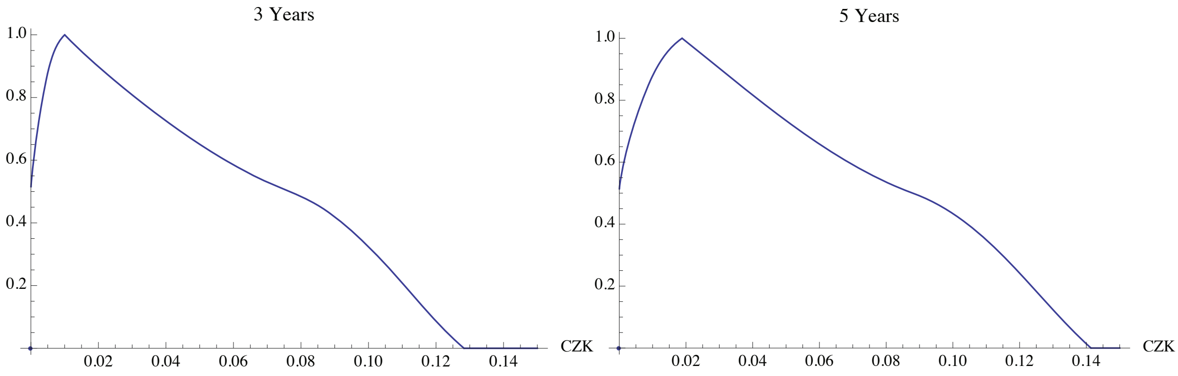
Having determined the fuzzy variance, the calculation of the fuzzy volatility can be finalized using a simple square root. The resulting fuzzy volatility constructed from fuzzy data can be written in bimatrix form as:

$$s_{LU}^{3} = \begin{pmatrix} (0, & 0), (0, & 0), (0.0100, & 0.0969) \\ (0.1283, & -0.1132), (0.0766, & -0.2385), (0.0100, & -0.0881) \end{pmatrix}, \tag{4.5}$$

$$s_{LU}^{5} = \begin{pmatrix} (0, & 0), (0, & 0), (0.0188, & 0.1329) \\ (0.1414, & -0.1084), (0.0880, & -0.2505), (0.0188, & -0.1162) \end{pmatrix}. \tag{4.6}$$

The graphical representation of the fuzzy volatility is shown in Figure 4.15. Both fuzzy volatilities are tilted to the left. The low membership degree of higher volatility values derives from the infrequent major fluctuations of the approximated fuzzy returns in the past. Although the shapes of both fuzzy volatilities are similar, the fuzzy volatility based on the five-year historical fuzzy returns series is wider, thus reaches higher values, due to major return fluctuations in 2008.

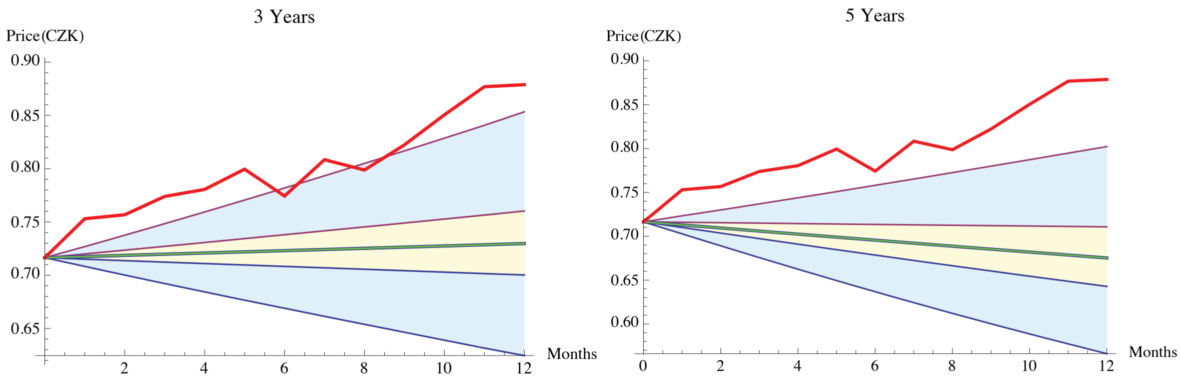
Figure 4.15 The Fuzzy Volatility



The Estimation of Price Evolution

The estimated fuzzy volatility can be now substituted into (3.22) to perform the estimation of the equity unit's price evolution in 2013. The input data are listed in Table 1, Table 2, (4.5) and (4.6). The estimated average price evolution, calculated as an arithmetic mean from all realizations using fuzzy arithmetic operations, and the real price evolution are shown in Figure 4.16. The approach to interpreting the α-cuts was explained in the subchapter 4.3.1.

Figure 4.16 Estimated Average Price Evolution Compared with the Real Price Evolution



As it can be seen from this figure, the estimated average price evolution based on the three-year historical fuzzy revenues series lies much closer to the real evolution than the approach based on the five-year fuzzy series. Great results have been obtained for June and August 2013, where the estimation was correct, and the real price has been included in the α-cut 0.2. The upper bound of the α-cut 0.2, which represents the optimistic price evolution, appears to be the best estimation of the equity unit's price evolution.

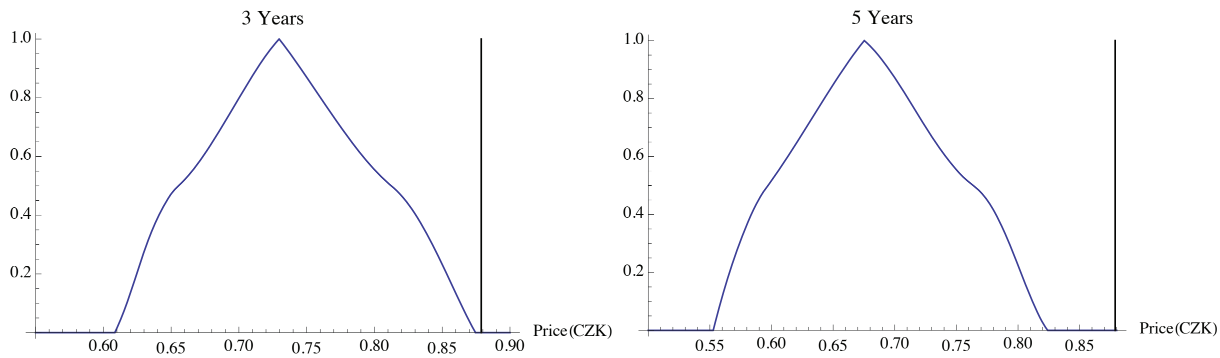
The Estimation of the Final Price

The estimated average final price of the equity unit and its real price as at December 2013 are shown in Figure 4.17.

The shapes of the fuzzy prices are very similar. The difference between these prices lies mainly in their range. The final fuzzy price calculated from the five-year fuzzy data series is wider, as was the case with the associated fuzzy volatilities, thus the possible range for the final price is bigger.

The α-cut 1 of the average final price based on the three-year fuzzy data is 0.7297 CZK, and since the initial price of the equity unit as at December 2012 was 0.7164 CZK according to the estimation the unit's price has increased. On the other hand the α-cut 1 of the average final price based on the five-year fuzzy data is 0.6753 CZK, therefore the estimation indicates a decrease of the price. Because the real price of the equity unit as at December 2013 was 0.8788 CZK it was proven again that by using the three-year historical data series the calculated estimation is more accurate.

Figure 4.17 The Estimated Average Final Price and the Real Price of the Equity Unit



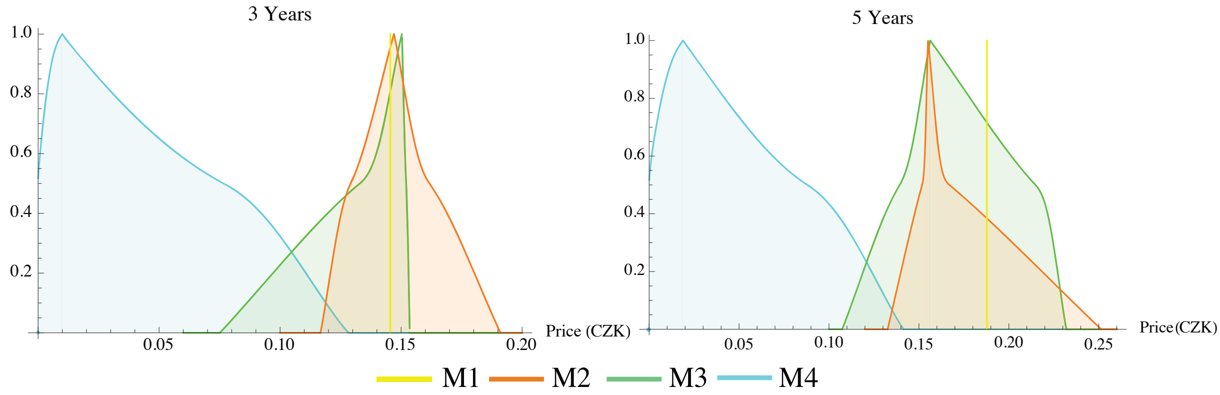
4.3.4 Assessment of the Results

The assessment of the results from the applied methods will be performed in the following text. The attention will be focused on the constructed volatilities and on the estimated price evolutions and final prices.

Assessment of Constructed Volatilities

The construction of volatility, or fuzzy volatility, has been performed using four different techniques. For a better understanding of the obtained results, all volatilities are presented in Figure 4.18.

Figure 4.18 Comparison of Fuzzy Volatilities



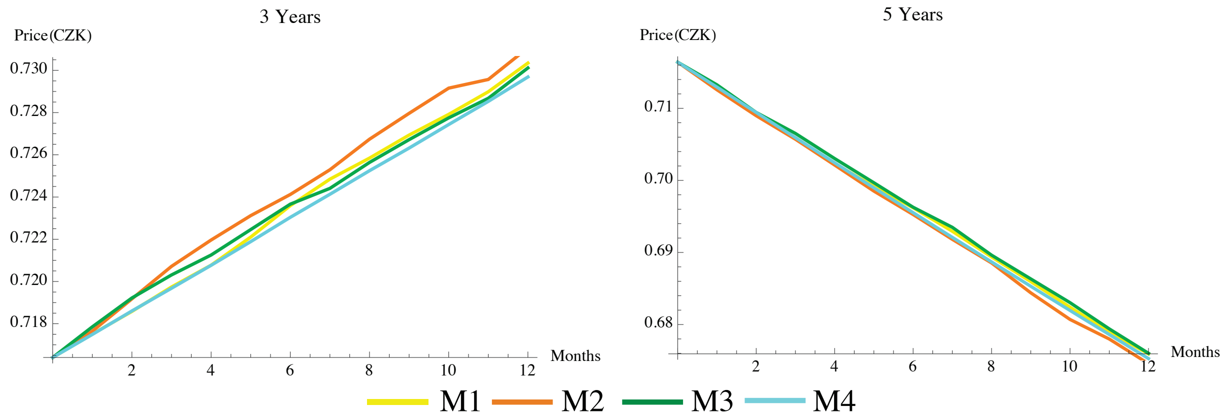
From this figure it is evident that the fuzzy volatility computed from fuzzy data (by M4) is much lower than the remaining volatilities and is the widest. The fuzzy volatility constructed by the M2 has incorporated the highest values of volatility. While focusing on the volatilities calculated from the three-year historical data series, the results from the M2 and M3 seem to have almost identical α-cut 1 which is also similar to the crisp volatility from the M1. This also applies to the results calculated from the five-year historical data series, but with slightly larger distances between these values.

It is obvious that by using fuzzy numbers it is possible to incorporate much more information about the past price evolution into the volatility parameter. From the results it appears that fuzzy volatilities, in particular the fuzzy volatility from the M4, represent the reality more accurately.

Assessment of the Estimated Price Evolutions

Further assessment of these volatilities will be performed in relation to the results of the price evolution estimations. The estimated average price evolutions represented by the α-cut 1, or by a crisp number from the M1, are depicted in Figure 4.19.

Figure 4.19 Comparison of the α-cut 1 and the Crisp Number of the Estimated Average Price Evolutions

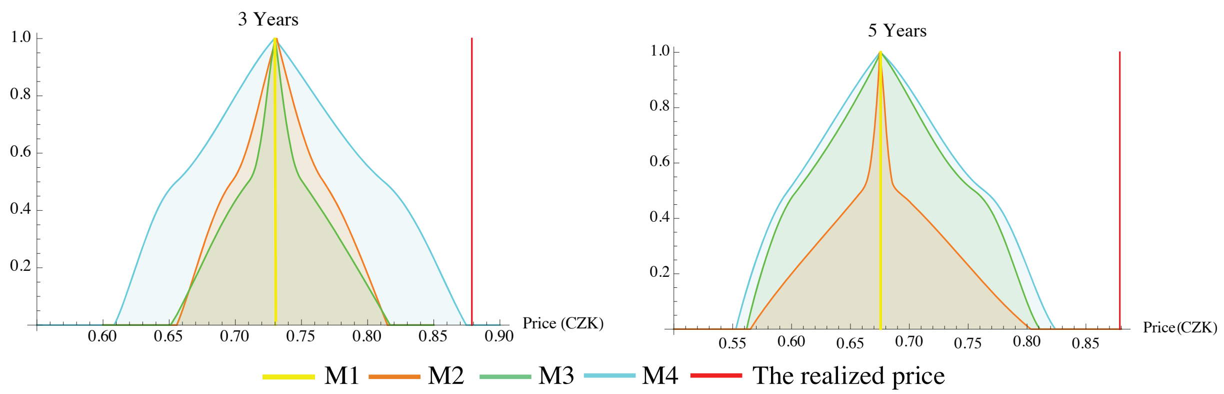


It is obvious that the Monte Carlo simulation with GBM applied with any technique of volatility construction leads to the same core of the estimated price evolution. The differences of the simulations results lie in the complexity of the resultant fuzzy numbers.

Assessment of the Estimated Final Prices

The graphical representation of the estimated average final prices of the equity unit along with its real price in December 2013 is shown in Figure 4.20.

Figure 4.20 Comparison of Estimated Average Final Prices and the Real Unit's Price



The estimated average price of the equity unit as at December 2013 was forecasted most accurately using the M4. This fuzzy price has the widest span which comes from the shape of the fuzzy volatility. This fact improves the estimation. Having an average value that contains this amount of information is very useful for the investor's decision-making process. Investors can incline towards the more optimistic or pessimistic side of the estimation according to their risk attitude. In the case of this investment instrument, an investor with a risk-loving attitude would increase his profits by taking into account information from this estimation.

It is interesting to see that the final prices calculated from the five-year historical data by the M3 and M4 have quite similar shapes. This shows that in some cases even the M3 approach can give extensive information. Also the base of the estimated price calculated by the M2 is very similar to the rest of the fuzzy numbers. But still, the estimation results calculated from the five-year source data series have been much less accurate, and have not portrayed the reality as well as the results from the estimation calculated from the three-year historical data series.

Since all methods will be further explored by their application on different investment instruments, the final assessment of the results will be at the end of the following subchapter.

4.4 Application of Selected Methods on Different Instruments

This subchapter will be focused on the application of selected methods on different investment instruments, considering only the three-year historical data time series. These instruments will be the *CZK/EUR FX* rate and *gold*.

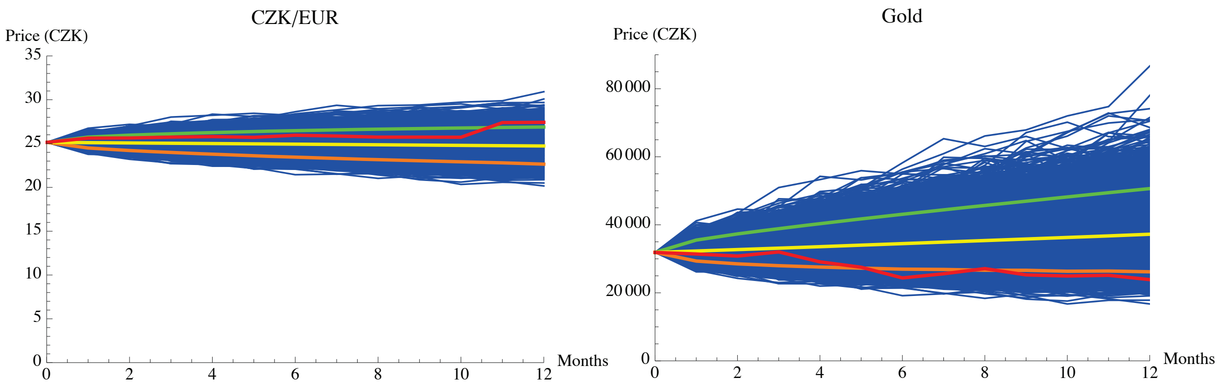
Since the procedures of each method will be the same as in the previous subchapters, only the results of applied methods will be discussed in this section. All calculations are based on the input data from Table 1 and Table 2.

4.4.1 Monte Carlo Simulation with GBM

The price evolution estimated for both the CZK/EUR FX rate and gold is shown in Figure 4.21. It can be seen that the estimated evolution of the FX rate has quite accurately incorporated real price evolution (red line). Only in November and December the price has exceed the 95th percentile represented by the green line. This is caused by the CNB's intervention which took place in November 2013 and led to an artificial depreciation of the FX rate. One could expect that this kind of intervention can be hardly predicted from the historical data. Still some of the realizations of the estimated rate evolution have approximated even this major change of the FX rate. The estimation was very accurate till November due to the low volatility (0.0521 CZK) which simplified the prediction in stable conditions.

The estimation of the gold price is mostly too optimistic, compared to the real price evolution. This is caused by the historical background of the returns which has outlined an increase of the price with a positive mean. Very accurate results have been obtained till March 2013 where the mean of the estimated prices has correctly approximated the real price evolution. However from April 2014 the real price evolution almost copied the 5th percentile represented by the orange line. It can be seen that the span of the predicted prices is wide. This is caused by a relatively high volatility (0.2005 CZK).

Figure 4.21 The Estimated and Real Price Evolution of Selected Investment Instruments

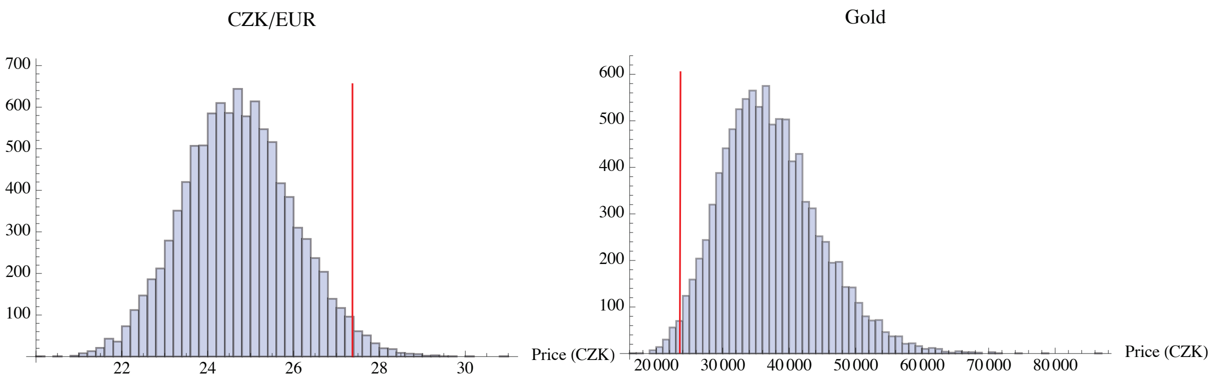


The Estimation of the Final Price

The histogram of estimated final prices from all 10 000 realizations is depicted in Figure 4.22. It can be seen that the majority of estimated final FX rates is lower than the real value of the FX rate. The mean of the estimated final rate is 24.7234 CZK/EUR, whereas the real value reached 27.425 CZK/EUR. The CNB's intervention caused an unexpected shift of the rate which was predicted with very low probability, due to the low volatility parameter.

The mean of the estimated final gold price for troy ounce is 37 229.5 CZK. The real price of that instrument as at December 2013 was much lower, that is 23 888.71 CZK. Even though the estimated average price evolution had an increasing trend, due to the returns positive mean (0.1534 CZK), the high value of the volatility allowed that the estimated final prices include the realized price (although associated with very low probability).

Figure 4.22 Histogram of Estimated Prices and the Real Price as at December 2013



4.4.2 Monte Carlo Simulation with GBM with Fuzzy Volatility

The approaches to price evolution estimation with the fuzzy volatility, that is the M2, M3 and M4, will be applied to the *CZK/EUR FX rate* and *gold* in this subchapter.

Fuzzy Volatility Constructed Using Box Approach

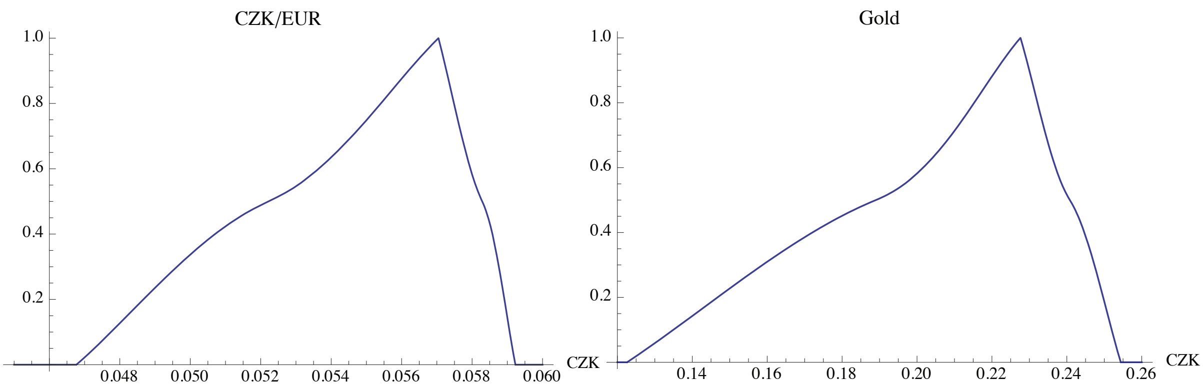
The first applied approach is the M2 from which the constructed fuzzy volatility of each investment instrument is graphically represented in Figure 4.23 and can be written as:

$$s_{LU}^{CZK} = \begin{pmatrix} (0.0468, \quad 0.0109), (0.0522, \quad 0.0206), (0.0570, \quad 0.0096) \\ (0.0592, -0.0019), (0.0583, -0.0044), (0.0570, -0.0025) \end{pmatrix}, \tag{4.7}$$

$$s_{LU}^{Gold} = \begin{pmatrix} (0.1226, \quad 0.1328), (0.1890, \quad 0.2101), (0.2276, \quad 0.0772) \\ (0.2544, -0.0272), (0.2408, -0.0536), (0.2276, -0.0264) \end{pmatrix}. \tag{4.8}$$

It can be seen that both fuzzy volatilities have very similar shapes. In both fuzzy numbers the membership degree 1 is assigned to a higher volatility value, i.e. the fuzzy numbers are tilted to the right. This is caused by the fact that in both data series only one partial volatility had significantly lower value.

Figure 4.23 The Fuzzy Volatility

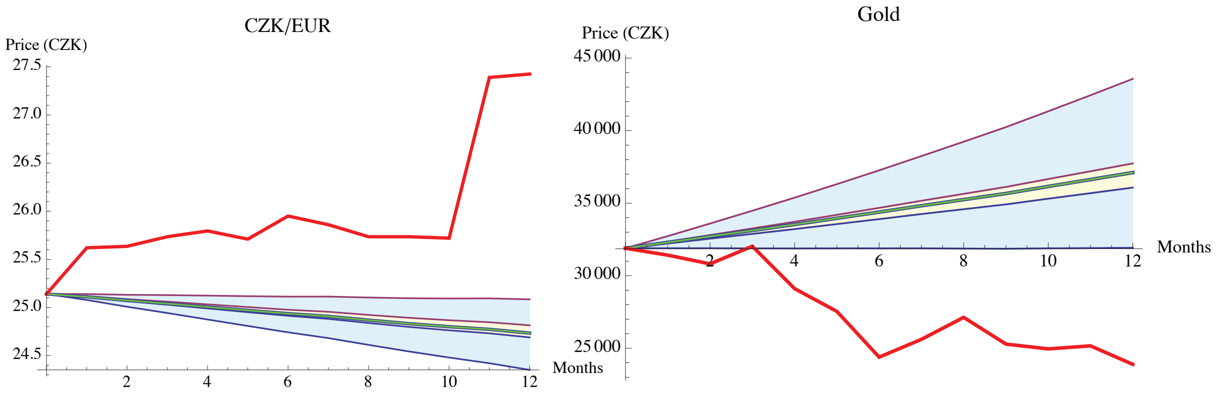


The Estimation of Price Evolution

The resulting mean of the estimated price evolution can be seen in Figure 4.24. The estimated evolution of CZK/EUR FX rate has a decreasing trend, whereas the real price has increased due to the CNB's intervention. It is obvious that the mean calculated from the historical data significantly affects the estimated evolution, and can lead to incorrect estimation when a rapid change of trend takes place. The width of the estimated rate evolution is relatively narrow, due to the low volatility.

Similar situation can be seen in the case of gold. Only till March 2013 the pessimistic (lower) bound of the α-cut 0.2 represented an appropriate approximation of the real price evolution. Even though the price evolution was expected to have an increasing trend, the real price has eventually decreased. This situation shows the importance of other investment instrument analyses that have to be performed when estimating the price evolution. It is essential to have more information about the current situation on financial markets that are not only based on hard but also soft data.

Figure 4.24 The Estimated and Real Price Evolution of Selected Investment Instruments



The Estimation of the Final Price

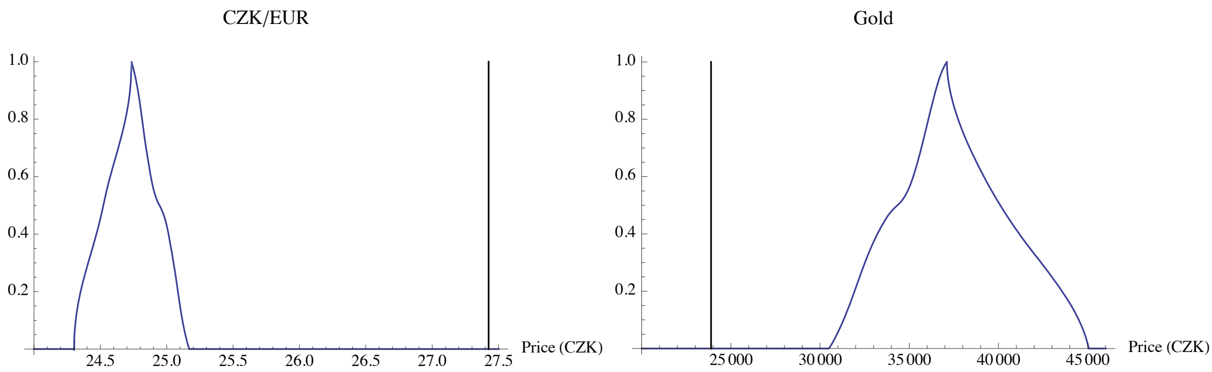
The mean values of the estimated final prices and the real prices of both investment instruments are shown in Figure 4.25.

It can be seen that the final fuzzy FX rate of CZK/EUR is very narrow. This shape comes from the fuzzy volatility of the FX rate. Analogous similarity can be seen between the fuzzy price of gold and the associated fuzzy volatility which is higher and has a wider span.

The estimated final value of the CZK/EUR FX rate (i.e. the α-cut 1) is 24.7338 CZK/EUR. Since the real price has reached 27.425 CZK/EUR it is obvious that this price has not been included in the estimated final fuzzy rate.

The same conclusion can be drawn from the estimation of the gold price. The estimated final price is 37 108.3 CZK whereas the final real price has reached 23 888.71 CZK. However, partly due to the relatively high and wide fuzzy volatility, the width of the final price is larger compared to the final price of the FX rate, therefore the final estimated price is closer to the real price of gold. Of course the accuracy of the estimation is also largely influenced by the applied mean.

Figure 4.25 The Estimated Average Final Price and the Real Price of the Instrument



Fuzzy Volatility Constructed Using Fuzzy Partition Approach

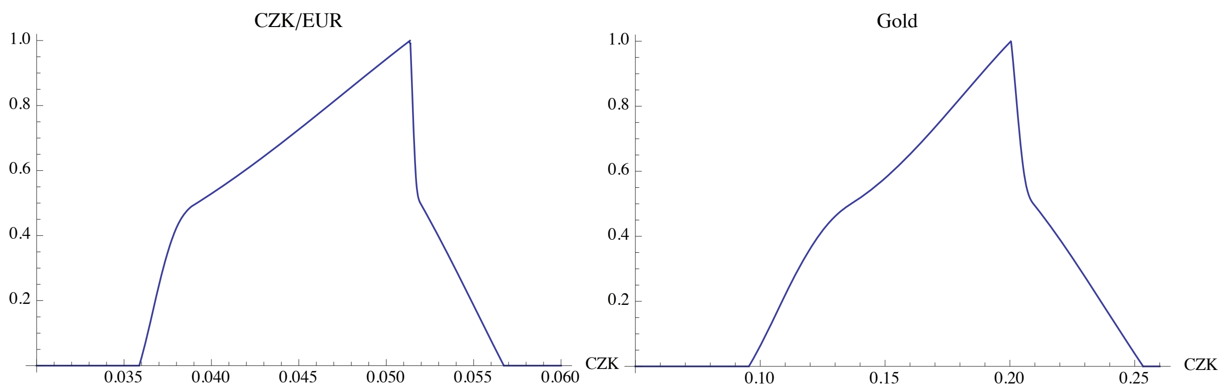
The following approach applied to the selected investment instruments is the M3. The resultant fuzzy volatilities calculated from the input data given in Table 1 are written in (4.9) and (4.10) differentiated by the investment instrument. Their graphical representation is shown in Figure 4.26.

$$s_{LU}^{CZK} = \begin{pmatrix} (0.0359, \quad 0.0065), (0.0391, \quad 0.0310), (0.0514, \quad 0.0245) \\ (0.0567, -0.0096), (0.0520, -0.0107), (0.0514, -0.0012) \end{pmatrix}, \tag{4.9}$$

$$s_{LU}^{Gold} = \begin{pmatrix} (0.0954, \quad 0.0820), (0.1364, \quad 0.2100), (0.2004, \quad 0.1280) \\ (0.2534, -0.0880), (0.2094, -0.1059), (0.2004, -0.0179) \end{pmatrix}. \tag{4.10}$$

The shapes of both fuzzy volatilities are very similar. The higher membership degrees are assigned to the higher values of volatility, i.e. the fuzzy numbers are tilted to the right. This is caused, in both cases, by the fact that three of the five partial volatilities have higher values. The lower part of the fuzzy numbers is symmetrical since the values of partial volatilities, excluding the partial volatility in α-cut 1, are relatively regularly distributed.

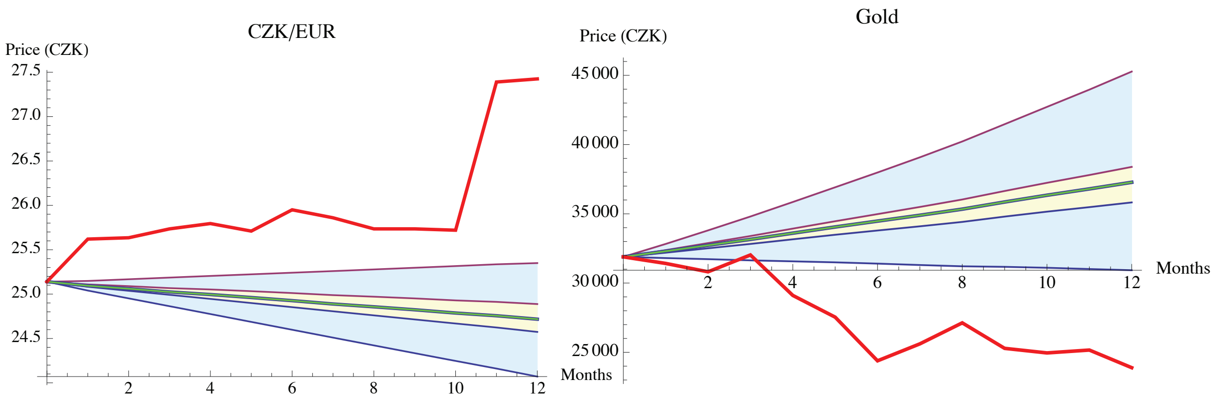
Figure 4.26 The Fuzzy Volatility



The Estimation of Price Evolution

The mean values of the estimated price evolution, calculated for both investment instruments, are shown in Figure 4.27. It is obvious that the estimated price evolution is not a correct approximation of the real price evolution. The results are similar to the one discussed in M2. The only difference can be seen in the width of the estimated fuzzy prices. While the fuzzy volatilities in M3 are wider compared to the fuzzy volatilities in M2, it resulted in wider estimated fuzzy prices. Therefore the estimation is closer to the real price evolution of investment instruments.

Figure 4.27 The Estimated and Real Price Evolution of Selected Investment Instruments



The Estimation of the Final Price

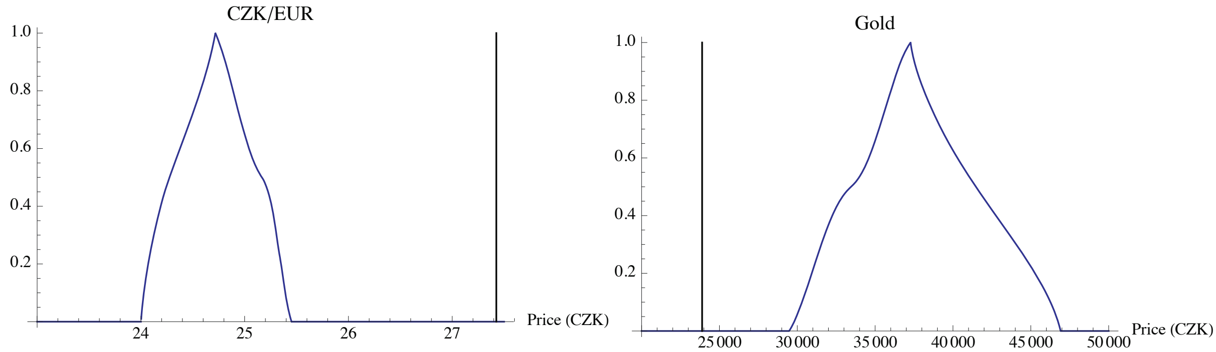
The mean values of the estimated final prices and the real prices of the investment instruments are depicted in Figure 4.28.

The width of both fuzzy prices corresponds with the shape of the fuzzy volatilities discussed above. It appears that the wider the fuzzy volatility, the wider the final fuzzy price.

The α-cut 1 of the final fuzzy FX rate of the CZK/EUR is 24.7175 CZK/EUR. Compared to the real price, which reached 27.425 CZK, it is obvious that the large shift of the rate in 2013 was not estimated. Even though it was partly caused by the low volatility, the change of the rate was abrupt (influenced by the CNB's intervention) therefore it was very difficult to predict this kind of evolution on the basis of historical data.

The α-cut 1 of the final fuzzy price of gold is 37 276.2 CZK, while the real price in December 2013 reached 23 888.71 CZK. In this case, the drop of the price was not estimated due to the positive mean that has outlined the positive trend of the estimated price evolution.

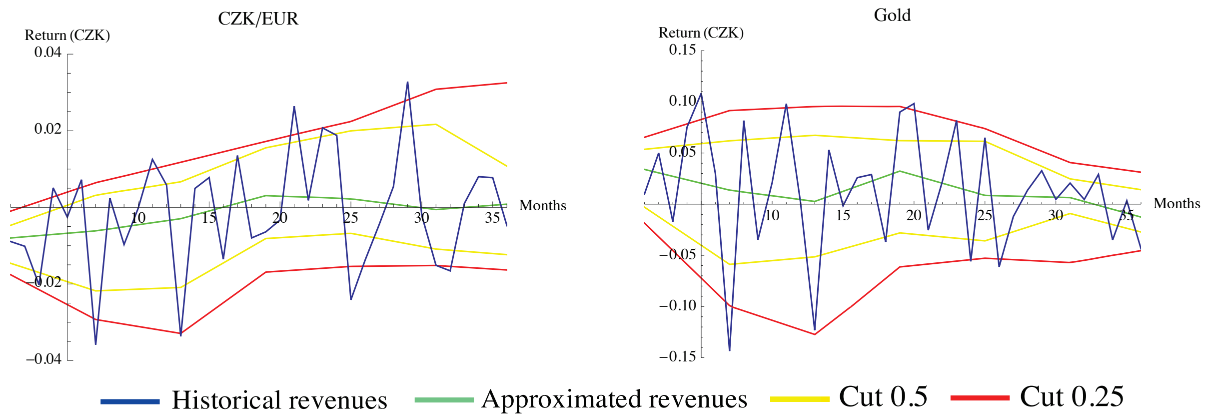
Figure 4.28 The Estimated Average Final Price and the Real Price of the Instrument



Fuzzy Volatility Constructed Using Fuzzy Transform Approach

The final, most complex approach, applied in this thesis is the M4. The first step of this technique is to create five approximated data time series which are calculated from the original historical returns time series. These approximated data series, represented by five cuts, are shown in Figure 4.29 differentiated by the type of an investment instrument.

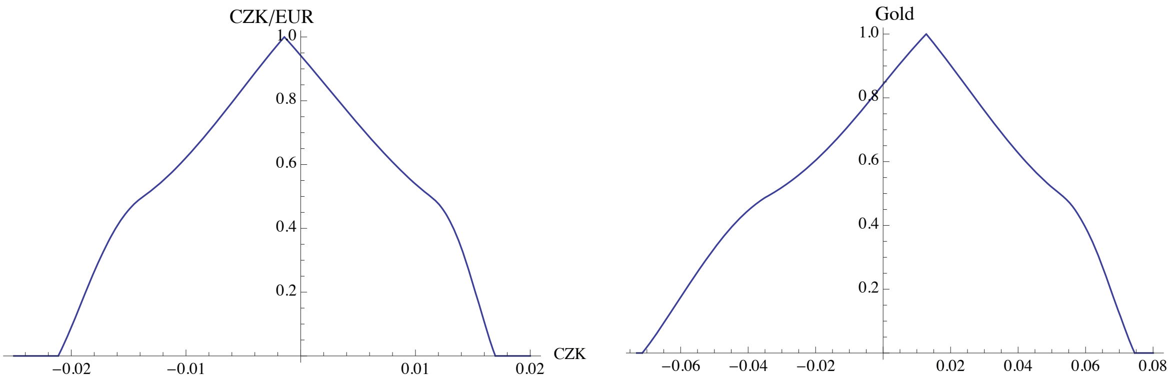
Figure 4.29 The Historical Returns Time Series with Five Approximated Cuts



The Fuzzy Volatility Construction Process

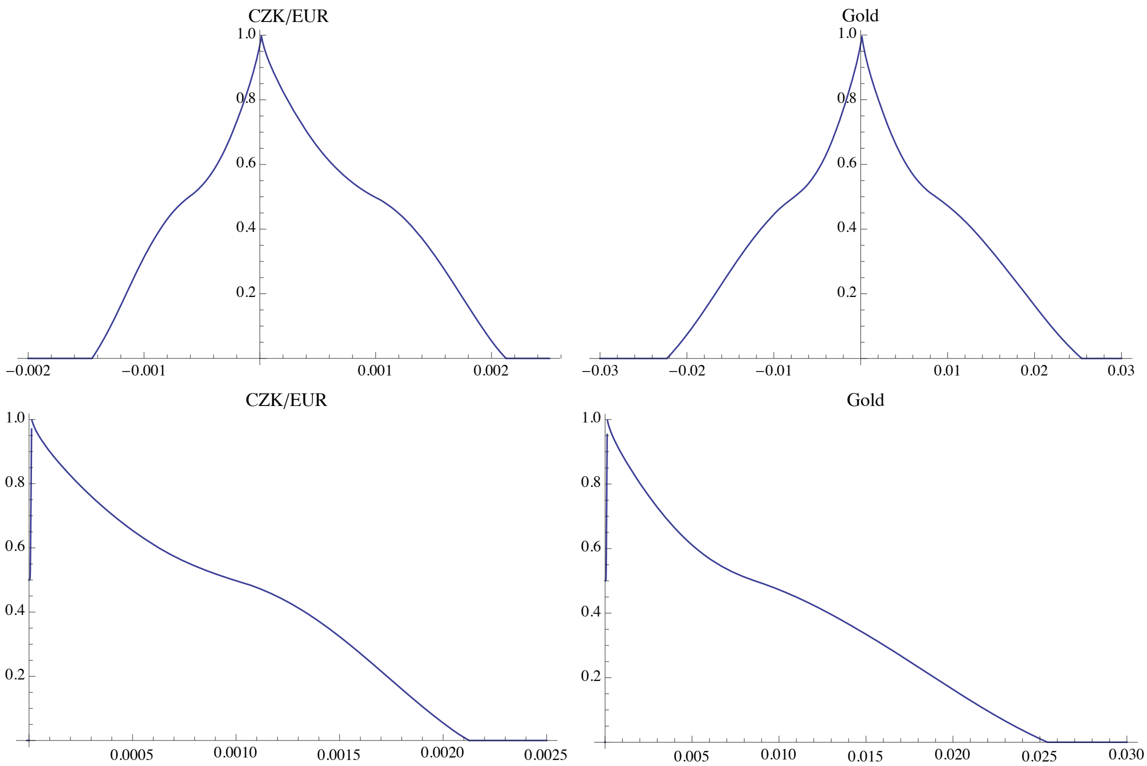
The fuzzy numbers, which represent the input data for this approach, are created from the approximated data series. The next step is to calculate the fuzzy mean which is shown in Figure 4.30. Both fuzzy mean numbers are concentrated near zero, where the mean for gold has higher values and indicates more significant increasing and decreasing trend. The fuzzy mean of the CZK/EUR FX rate is tilted slightly to the left, thus indicates the decreasing trend of the estimated price evolution. The fuzzy mean of gold is tilted to the right therefore the increasing trend of the estimated price evolution can be expected.

Figure 4.30 The Fuzzy Mean



Having determined the fuzzy mean, the next step is to calculate the fuzzy variance. This step has to precede the fuzzy volatility calculation, because part of the fuzzy variance has negative values, and the positive part of the fuzzy variance has to be applied. The original fuzzy variance and the positive part of fuzzy variance are depicted in Figure 4.31.

Figure 4.31 The Fuzzy Variance and The Positive Part of Fuzzy Variance



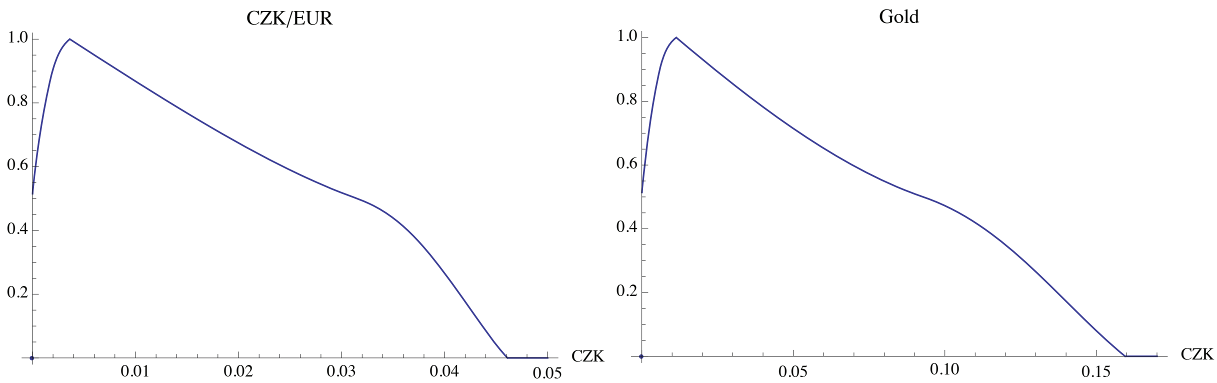
Having adjusted the fuzzy variance, the fuzzy volatility can be calculated by a square root. The resultant values of fuzzy volatilities are presented in (4.11) and (4.12) and their graphical representation is shown in Figure 4.32.

$$s_{LU}^{CZK} = \begin{pmatrix} (0, & 0), (0, & 0), (0.0036, & 0.0386) \\ (0.0461, -0.0298), & (0.0315, -0.0853), & (0.0036, -0.0478) \end{pmatrix}, \quad (4.11)$$

$$s_{LU}^{Gold} = \begin{pmatrix} (0, & 0), (0, & 0), (0.0114, & 0.1429) \\ (0.1595, -0.1345), & (0.0928, -0.3024), & (0.0114, -0.1206) \end{pmatrix}. \quad (4.12)$$

From the figure it can be seen that the shapes of fuzzy volatilities are almost identical (both tilted to the left). The low membership degree of higher volatility values comes from the infrequent major fuzzy revenues fluctuations. Although the fuzzy volatilities have similar shapes, the fuzzy volatility of gold returns is much wider, thus incorporates higher volatility values.

Figure 4.32 The Fuzzy Volatility

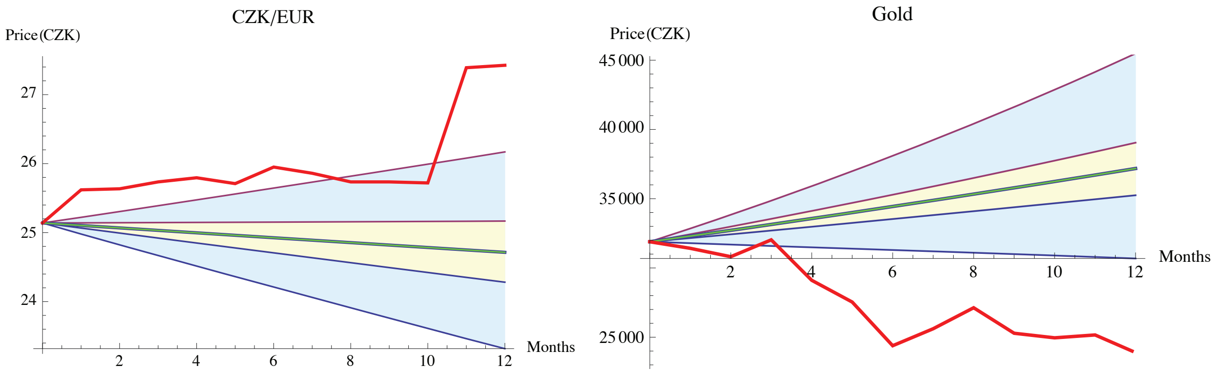


The Estimation of Price Evolution

The estimated mean of the fuzzy price evolution can be seen in Figure 4.33. The real evolution of the CZK/EUR FX rate has been included in α-cut 0.2 from August to October 2013. This accurate estimation was obtained even though the estimation has an opposite trend to the real rate evolution.

In the case of gold, accurate estimation was obtained only in March, where the real price was incorporated in the α-cut 0.2 of the estimated fuzzy price. As it was mentioned earlier, the positive mean of gold returns had affected the trend of the estimation which does not correspond with the real price evolution in 2013.

Figure 4.33 The Mean of the Estimated Price Evolution and the Real Price Evolution



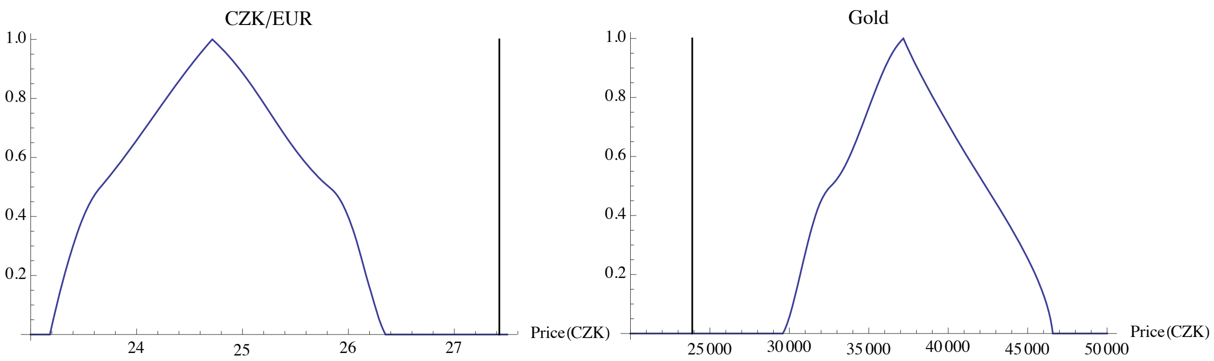
The Estimation of the Final Price

The mean values of the estimated final prices of the investment instruments and their real prices in December 2013 are presented in Figure 4.34.

The mean value from all estimated final CZK/EUR FX rates is 24.7142 CZK/EUR. Since the real price of the FX rate in December 2013 reached 27.425 CZK/EUR, the α-cut 1 has not portrayed the real rate evolution accurately. On the other hand the complexity of the final fuzzy rate helped to approximate the real price evolution by the optimistic (upper) bound of the α-cut 0.2.

A similar situation can be seen in the estimated final price of gold, where the pessimistic (lower) bound is closest to the real price of gold. In this case the α-cut 1 of the estimated final fuzzy price is 37 179.7 CZK and the real price reached 23 888.71 CZK in December 2013. It is obvious that the positive value of the mean had negatively influenced the accuracy of the estimation, since the real price evolution had had the opposite trend.

Figure 4.34 The Estimated Average Final Price and the Real Price of the Instrument



4.4.3 Assessment of the Obtained Results

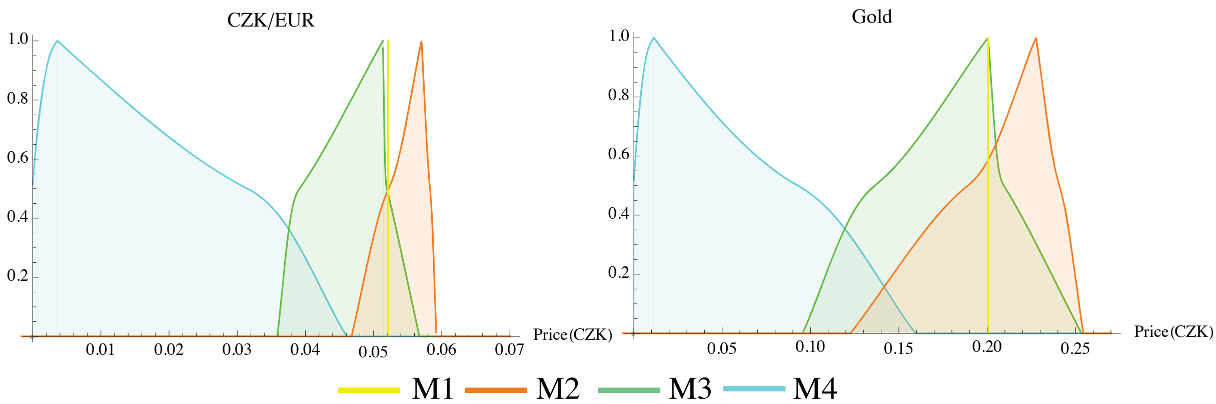
The assessment of the obtained results from the methods applied to the CZK/EUR FX rate and gold will be discussed in this subchapter. The attention will be focused on three main areas: the constructed volatilities, the estimated price evolutions and the estimated final prices.

Assessment of Constructed Volatilities

The comparison of all constructed fuzzy volatilities is shown in Figure 4.35 differentiated by the investment instrument. Even though the fuzzy volatility calculated by M4 has in both cases the lowest values, this fuzzy number contains the widest range of volatilities. Particularly in the case of CZK/EUR FX rate, the widths of the remaining two fuzzy volatilities are significantly narrower compared to the fuzzy volatility constructed by the M4. On the other hand the fuzzy volatility computed by M2 contains the highest values of volatility. For both investment instruments, the fuzzy volatilities computed by M2 and M3 have similar values in the α-cuts 1 and are also comparable with the crisp volatility calculated by M1.

By using the fuzzy volatilities it was possible to incorporate much valuable information into the volatility parameter. It can be already said that the crisp volatility will not be the most appropriate volatility parameter, since many interesting characteristics of the returns fluctuations tend to disappear. Further assessment of the volatilities (fuzzy volatilities), will be performed to establish the most appropriate construction method.

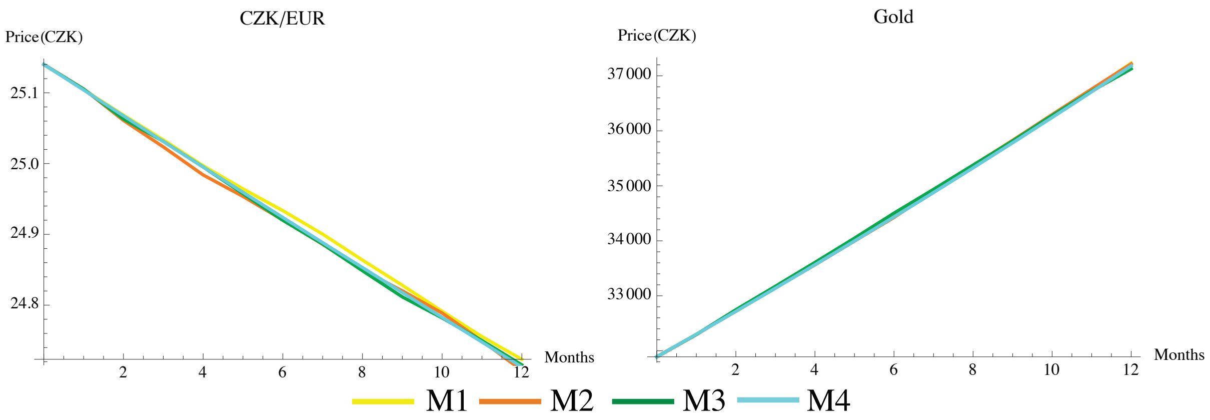
Figure 4.35 Comparison of Constructed Volatilities



Assessment of the Estimated Price Evolutions

The estimated average price evolutions represented by the α-cut 1 of the fuzzy prices and by a crisp number in the case of M1 are shown in Figure 4.36. It can be seen that these prices form a nearly identical line, thus it was confirmed that the results of the Monte Carlo simulation with GBM performed with any approach of volatility construction are almost identical in the core of the estimated prices. As it was mentioned earlier, the differences of the simulations' results can be found in the complexity of the resultant fuzzy numbers.

Figure 4.36 Comparison of the α-cut 1 and the Crisp Number of the Estimated Price Evolution



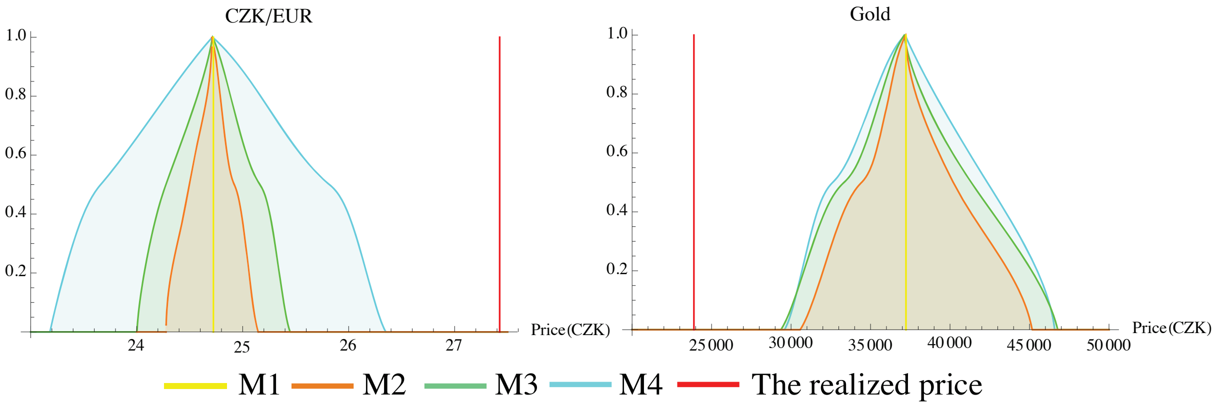
Assessment of the Estimated Final Prices

The final comparison is focused on the estimated average final prices of both instruments along with their real prices in December 2013 which are presented in Figure 4.37.

In the case of CZK/EUR FX rate, it is obvious that the M4's resulting estimation is the best approximation of the real price. Compared with the remaining results, the optimistic (upper) bound is the closest to the realized price. It is interesting to observe how a wider fuzzy volatility, even with lower values, affects the estimated price; it broadens the fuzzy price and gives us more information and more realistic estimation of the future price.

Nevertheless the estimated final fuzzy prices of gold are much more similar. This situation can occur when the constructed fuzzy volatilities are comparable. This is particularly obvious in the case of fuzzy volatilities and fuzzy prices constructed by M3 and M4. Still the M4's fuzzy volatility and final fuzzy price are the largest and contain the most information.

Figure 4.37 Comparison of Estimated Average Final Prices and the Real Price



Despite the fact that all methods were calculated from the three-year historical data series, the same calculations based on the five-year historical data series were performed beyond the diploma thesis. The estimated price evolution for both investment instruments had the same trend as the estimations in this thesis. The main difference in the results of all methods regarding both instruments was spotted in the higher values of the calculated volatilities and fuzzy volatilities. The results for gold have confirmed the discussed estimations. The only difference was spotted in the width of the estimated fuzzy prices which were slightly narrower or wider depending on the applied approach, however the accuracy of the estimations was comparable. In the case of the CZK/EUR FX rate, the five-year historical data series appear to be more appropriate for its price evolution estimation. The estimated fuzzy prices always had a wider span, thanks to which the estimations were more accurate. This fact was caused by the major change of the FX rate in 2008, due to which the fuzzy volatility had higher values resulting in wider estimated fuzzy prices.

Final assessment of selected methods

Having assessed all obtained results, the most suitable approach of volatility construction appears to be the M4. When applied to various investment instruments and different historical data time series, the constructed fuzzy volatility substituted into the Monte Carlo simulation with GBM allowed to obtain the most accurate results which were validated by the real price evolution. It was shown that the application of fuzzy numbers in price estimation helps to model the vague and complex financial reality more accurately. The fuzzy volatility and the final fuzzy price can be interpreted in different ways depending on the investor's risk attitude. This diversity is a great advantage of this approach. Having a single fuzzy number that contains that much information is very useful in the complex financial reality. Not only the investment decision-making process can be improved, but it can also help in the comparison of different investment instruments. However, it is important to remember that the estimated fuzzy price has to be interpreted on the basis of additional information provided from other investment instruments analysis.

In summary, it was determined that the M4 approach appears to be the most effective tool for estimation of the price evolution which, combined with other investment instrument analyses, can significantly improve the investment decision-making and therefore lead to higher profits.

5 Conclusion

The objective of this diploma thesis was the assessment of alternative methods for volatility estimation of selected investment instruments which were applied to the Monte Carlo simulation with geometric Brownian motion.

In the first chapter of the thesis, the investment, investment instruments, financial systems and the players that engage in the financial markets were characterised. There was also described the investment process, criteria for investment decisions and financial models.

In the second chapter, the attention was focused on the theoretical basis for the volatility construction and the estimation of price evolution. Four approaches of volatility construction were described, namely, *the crisp volatility (M1)*, *the box approach (M2)*, *the fuzzy partition approach (M3)* and *the fuzzy transform approach (M4)*. In this section also the Monte Carlo simulation with GBM, or with fuzzy volatility, was described.

In the third chapter of this thesis, all methods described in the second chapter were applied to three different investment instruments and with the calculations based on two different ranges of historical data time series. In conclusion, the assessment of all obtained results was performed.

It was determined that approaches of fuzzy volatility construction are more appropriate for the estimation of the price evolution than the crisp volatility calculation, since the fuzzy numbers contain much more information. When using the crisp volatility, many interesting characteristics of volatility tend to disappear. While using the fuzzy numbers, the historical volatility can be described more precisely which improves the results of price estimations. Also the estimated price in the form of a fuzzy number provides much more information about the unknown future price which can help investors to make more serious investment decisions. Not only the investment decision-making process can be improved, but it can also help in the comparison of different investment instruments. Investors can incline towards the more optimistic or pessimistic side of estimated values based on their risk attitude. This diversity is a great advantage of fuzzy numbers in the complex financial reality.

While comparing the constructed fuzzy volatilities it was observed that the α-cuts 1 of the fuzzy volatilities constructed by the M2 and M3 are very similar, and are also comparable to the crisp volatility. The fuzzy volatilities constructed by the M2 have also the highest values. On the other hand the fuzzy volatility constructed by the M4 is distinctly wider than the remaining fuzzy volatilities and has the lowest values.

Regarding the estimation of the price evolution, it was determined that the estimated price evolutions calculated by the Monte Carlo simulation with GBM performed with any approach of volatility construction have almost identical α-cut 1, and are also comparable to the estimated price evolution by M1. The differences of the simulations results can be found in the complexity of the resultant fuzzy prices.

The most accurate estimation calculated in this diploma thesis was computed by M4 from the three-year fuzzy returns series of the CZK/EUR FX rate where three real price values (from August to October 2013) have been included in the estimated average fuzzy prices (in the α-cut 0.2). In the case of the remaining two instruments, the most accurate results were obtained also by the M4.

It was observed that the selection of the range of the historical data time series significantly influences the accuracy of the estimation. The most accurate results of the estimations of the price evolutions for the equity unit and gold were calculated on the basis of the three-year historical data series. Only in the case of CZK/EUR FX rate more accurate estimations would be obtained from the five-year historical data series.

It should be kept in mind that the price estimation represents an additional information source and has to be assessed in relation with other investment instrument analyses. In this diploma thesis it was shown that having quality information from other analyses, the estimation of price evolution calculated by M4 represents the most appropriate tool which can help to increase investor’s profits.

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List of Abbreviations

AVMC	antithetic variates Monte Carlo
CNB	Czech National Bank
CZK	Czech koruna
e.g.	for example
EUR	euro
FX	foreign exchange
GBM	geometric Brownian motion
i.e.	that is
IPS	Investment policy statement
M1	Monte Carlo simulation with GBM with the crisp volatility
M2	Monte Carlo simulation with GBM with the fuzzy volatility constructed by the box approach
M3	Monte Carlo simulation with GBM with the fuzzy volatility constructed by the fuzzy partition approach
M4	Monte Carlo simulation with GBM with the fuzzy volatility constructed by the fuzzy transform approach
p.a.	per annum
QMC	Quasi-Monte Carlo simulation
SSMC	stratified sampling Monte Carlo
μ	population mean
$\bar{x}$	sample mean
σ	population standard deviation
σ^2	population variance

c_i	central node
h	bandwidth
K	triangle generating function
n	number of realizations
N	number of steps
$\mathbb{R}$	set of real numbers
S	investment instruments price
S_T	final estimated price of an investment instrument
s	volatility (sample standard deviation)
s_{LU}	fuzzy volatility
s^2	sample variance
s_{LU}^2	fuzzy sample variance
Δt	time interval
ε	random variable
u	fuzzy number
δu	slope in fuzzy number
u^+	right endpoint of the α-cut
u^-	left endpoint of the α-cut
y_{t_j}	original datum
$\hat{y}_{t_j}$	approximated datum

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Patrycja Czudek
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Annex no. 1 Monthly Prices and Returns of the Equity Unit

Date	Price (CZK)	Return	Date	Price (CZK)	Return
21.12.2012	0.7164	0.0104	30.06.2010	0.6264	-0.0350
30.11.2012	0.7090	0.0099	31.05.2010	0.6487	-0.0735
31.10.2012	0.7020	-0.0155	30.04.2010	0.6982	0.0036
27.09.2012	0.7130	0.0227	31.03.2010	0.6957	0.0527
31.08.2012	0.6970	0.0299	26.02.2010	0.6600	0.0182
31.07.2012	0.6765	0.0089	29.01.2010	0.6481	-0.0457
29.06.2012	0.6705	0.0427	30.12.2009	0.6784	0.0356
31.05.2012	0.6425	-0.0824	30.11.2009	0.6547	0.0345
30.04.2012	0.6977	-0.0246	30.10.2009	0.6325	-0.0251
30.03.2012	0.7151	0.0074	30.09.2009	0.6486	0.0291
29.02.2012	0.7098	0.0510	31.08.2009	0.6300	0.0352
31.01.2012	0.6745	0.0500	31.07.2009	0.6082	0.0572
28.12.2011	0.6416	-0.0110	30.06.2009	0.5744	-0.0147
30.11.2011	0.6487	-0.0127	29.05.2009	0.5829	0.0616
31.10.2011	0.6570	0.0830	30.04.2009	0.5481	0.0728
30.09.2011	0.6047	-0.0772	31.03.2009	0.5096	0.0678
31.08.2011	0.6532	-0.0765	27.02.2009	0.4762	-0.0764
29.08.2011	0.7051	-0.0240	30.01.2009	0.5140	-0.0284
30.06.2011	0.7222	-0.0246	29.12.2008	0.5288	-0.0182
31.05.2011	0.7402	-0.0085	28.11.2008	0.5385	-0.0581
29.04.2011	0.7465	0.0062	31.10.2008	0.5707	-0.1999
31.03.2011	0.7419	-0.0154	30.09.2008	0.6970	-0.1399
28.02.2011	0.7534	0.0236	29.08.2008	0.8017	0.0043
31.01.2011	0.7358	0.0100	31.07.2008	0.7983	-0.0340
28.12.2010	0.7285	0.0636	30.06.2008	0.8259	-0.0833
30.11.2010	0.6836	-0.0154	30.05.2008	0.8976	0.0057
29.10.2010	0.6942	0.0319	30.04.2008	0.8925	0.0572
30.09.2010	0.6724	0.0513	31.03.2008	0.8429	-0.0384
31.08.2010	0.6388	-0.0373	29.02.2008	0.8759	-0.0084
30.07.2010	0.6631	0.0569	31.01.2008	0.8833	-0.0883
			31.12.2007	0.9648	-

Annex no. 2 Monthly Rates and Returns of the CZK/EUR

Date	Price (CZK)	Return	Date	Price (CZK)	Return
31.12.2012	25.140	-0.0048	30.06.2011	24.345	-0.0080
30.11.2012	25.260	0.0077	31.05.2011	24.540	0.0135
31.10.2012	25.065	0.0080	29.04.2011	24.210	-0.0135
27.09.2012	24.865	0.0010	31.03.2011	24.540	0.0078
31.08.2012	24.840	-0.0166	28.02.2011	24.350	0.0049
31.07.2012	25.255	-0.0151	31.01.2011	24.230	-0.0337
29.06.2012	25.640	-0.0021	31.12.2010	25.060	0.0058
31.05.2012	25.695	0.0328	30.11.2010	24.915	0.0125
30.04.2012	24.865	0.0054	29.10.2010	24.605	-0.0002
30.03.2012	24.730	-0.0044	30.09.2010	24.610	-0.0097
29.02.2012	24.840	-0.0138	31.08.2010	24.850	0.0024
31.01.2012	25.185	-0.0241	30.07.2010	24.790	-0.0359
30.12.2011	25.800	0.0188	30.06.2010	25.695	0.0072
30.11.2011	25.320	0.0208	31.05.2010	25.510	-0.0025
31.10.2011	24.800	0.0018	30.04.2010	25.575	0.0051
30.09.2011	24.755	0.0264	31.03.2010	25.445	-0.0202
31.08.2011	24.110	-0.0033	26.02.2010	25.965	-0.0102
29.07.2011	24.190	-0.0064	29.01.2010	26.230	-0.0089
			31.12.2009	26.465	-

Annex no. 3 Monthly Prices and Returns of Gold

Date	Price (CZK)	Return	Date	Price (CZK)	Return
31.12.2012	31 890.45	-0.0436	30.06.2011	25 259.08	-0.0368
30.11.2012	33 310.30	0.0034	31.05.2011	26 206.54	0.0290
31.10.2012	33 196.16	-0.0349	29.04.2011	25 456.25	0.0259
27.09.2012	34 374.75	0.0291	31.03.2011	24 805.46	-0.0016
31.08.2012	33 387.56	0.0046	28.02.2011	24 844.86	0.0531
31.07.2012	33 233.18	0.0206	31.01.2011	23 560.46	-0.1231
29.06.2012	32 555.20	0.0047	31.12.2010	26 647.05	0.0017
31.05.2012	32 401.39	0.0327	30.11.2010	26 601.02	0.0982
30.04.2012	31 358.31	0.0136	29.10.2010	24 113.48	0.0210
30.03.2012	30 935.63	-0.0121	30.09.2010	23 612.75	-0.0347
29.02.2012	31 312.67	-0.0612	31.08.2010	24 446.72	0.0817
31.01.2012	33 287.54	0.0650	30.07.2010	22 528.43	-0.1435
30.12.2011	31 194.14	-0.0559	30.06.2010	26 005.70	0.0298
30.11.2011	32 989.11	0.0818	31.05.2010	25 242.20	0.1083
31.10.2011	30 398.94	0.0223	30.04.2010	22 651.99	0.0752
30.09.2011	29 729.21	-0.0254	31.03.2010	21 011.4	-0.0170
31.08.2011	30 493.82	0.0983	26.02.2010	21 371.24	0.0500
29.07.2011	27 637.82	0.0900	29.01.2010	20 329.64	0.0099
			31.12.2009	20 129.49	-

Annex no. 4 Real Price Evolution of the Equity Unit, CZK/EUR, and Gold in 2013

Date	Equity Unit's Price (CZK)	CZK/EUR	Gold Price (CZK)
31.01.2013	0.7530	25.620	31 416.79
28.02.2013	0.7567	25.635	30 812.40
29.03.2013	0.7739	25.735	32 028.89
30.04.2013	0.7805	25.795	29 108.15
31.05.2013	0.7995	25.710	27 538.22
28.06.2013	0.7743	25.950	24 384.59
31.07.2013	0.8084	25.860	25 610.06
30.08.2013	0.7987	25.735	27 125.78
30.09.2013	0.8223	25.735	25 284.66
31.10.2013	0.8505	25.720	24 954.39
29.11.2013	0.8769	27.390	25 161.80
31.12.2013	0.8788*	27.425	23 888.71

*Note: * 23.12.2013*